Section 7.1 **The Law of Sines**

Law of Sines

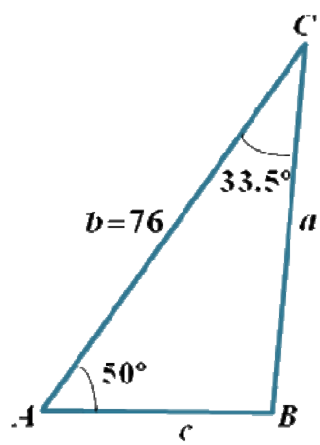
If A, B, and C are the measures of the angles of a triangle, and a, b, and c are the lengths of the sides opposite these angles, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The ratio of the length of the side of any triangle to the sine of the angle opposite that side is the same for all three sides of the triangle.

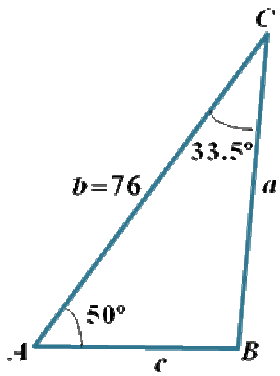
Example 92 Solve triangle ABC if $A = 50^\circ$, $C = 33.5^\circ$, and $b = 76$.

Solution We begin by drawing a picture of triangle ABC and labeling it with the given information. The figure shows the triangle that we must solve. We begin by finding B



$A + B + C = 180^\circ$	**The sum of the measurements of a triangle's interior angles is 180°.**
$50^\circ + B + 33.5^\circ = 180^\circ$	**$A = 50^\circ$ and $C = 33.5^\circ$.**
$83.5^\circ + B = 180^\circ$	**Add.**
$B = 96.5^\circ$	**Subtract 83.5° from both sides.**

Keep in mind that we must be given one of the three ratios to apply the Law of Sines. In this example, we are given that $b = 76$ and we found that $B = 96.5^\circ$. Thus, we use the ratio ${}^{b}/\sin B$, or ${}^{76}/\sin 96.5^\circ$, to find the other two sides. Use the Law of Sines to find a and c.



Find a:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 50^\circ} = \frac{76}{\sin 96.5^\circ}$$

$$a = \frac{76 \sin 50^\circ}{\sin 96.5^\circ} \approx 59$$

This is the known ratio.

Find c:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 33.5^\circ} = \frac{76}{\sin 96.5^\circ}$$

$$c = \frac{76 \sin 33.5^\circ}{\sin 96.5^\circ} \approx 42$$

The solution is $B = 96.5^\circ$, $a = 59$, and $c = 42$.

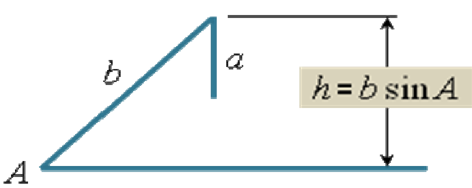
Example 93 Solve a triangle with A = 46°, C = 63°, and c = 56 inches.

The Ambiguous Case (SSA)

Consider a triangle in which *a*, *b*, and *A* are given. This information may result in:

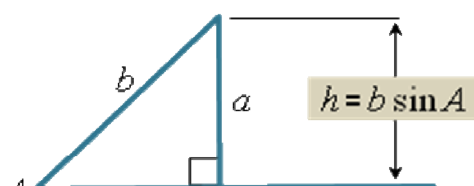
No Triangle

a is less than h and not long enough to form a triangle.



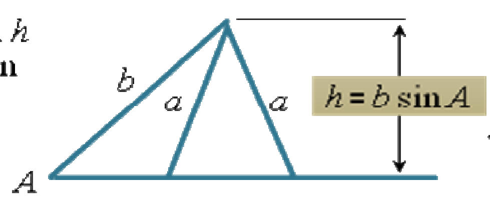
One Right Triangle

$a = h$ and is just the right length to form a right triangle.



Two Triangles

a is greater than h and a is less than b. Two distinct triangles are formed.

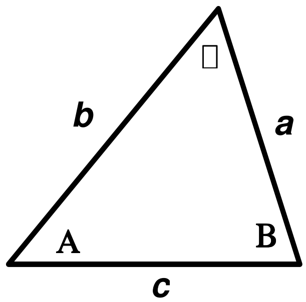


One Triangle

a is greater than h and a is greater than b. One triangle is formed.

Example 94

Solve the triangle shown with A=36º, B=88º and c=29 feet.



$$\frac{\sin 36}{a} = \frac{\sin 88}{b} = \frac{\sin 56}{29}$$

$$\frac{.59}{a} = \frac{1}{b} = \frac{.83}{29}$$

$$\frac{1}{b} = \frac{.83}{29} \quad so \quad .83b = 29 \qquad b = 34.94$$

$$\frac{.59}{a} = \frac{.83}{29} \quad so \quad .83a = 17.11 \qquad a = 20.61$$

Example 95 (no solution)

Solve triangle ABC is A = 75°, a = 51, and b = 71.

Example 96 (two solutions)

Solve triangle ABC is A = 40°, a = 54, and b = 62.

Area of An Oblique Triangle

The area of a triangle equals one-half the product of the lengths of two sides times the sine of their included angle. In the following figure, this wording can be expressed by the formulas:

$$Area = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

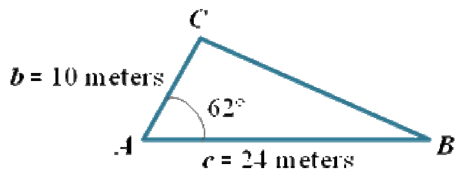
Example 97

Find the area of a triangle having two sides of lengths 24 meters and 10 meters and an included angle of 62º

Solution The triangle is shown in the following figure. Its area is half the product of the lengths of the two sides times the sine of the included angle.

$$\text{Area} = 1/2(24)(10)(\sin 62^\circ) = 106$$

the area of the triangle is approximately 106 square meters.



Example 98

Find the area of a triangle having two sides of lengths 12 ft. and 20 ft. and an included angle of 57º.

Solution:

$$Area = \frac{1}{2}bc\sin A = \frac{1}{2}(12)(20)\sin 57$$
$$= 120 * .84 = 100.8 sq.ft.$$

Section 7.2 **The Law of Cosines**

If A, B, and C are the measures of the angles of a triangle, and a, b, and c are the lengths of the sides opposite these angles, then

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

The square of a side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.

Solving an SAS Triangle

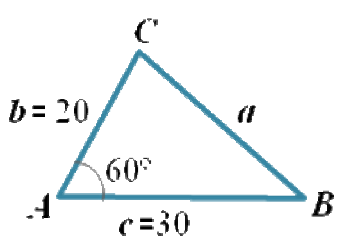
1. Use the Law of Cosines to find the side opposite the given angle.

2. Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute.

3. Find the third angle. Subtract the measure of the given angle and the angle found in step 2 from 180º.

Example 99

Solve the triangle shown with A = 60º, b = 20, and c = 30.



Step 1 **Use the Law of Cosines to find the side opposite the given angle.** Thus, we will find a.

$a^2 = b^2 + c^2 - 2bc\cos A$ Apply the Law of Cosines to find a.

$a^2 = 20^2 + 30^2 - 2(20)(30)\cos 60°$ $b = 20$, $c = 30$, and $A = 60°$.

$= 400 + 900 - 1200(0.5) = 700$ Perform the indicated operations.

$a = \sqrt{700} = 26$ Take the square root of both sides and solve for *a*.

Step 2 **Use the Law of Sines to find the angle opposite the shorter of the two given sides. This angle is always acute.** The shorter of the two given sides is $b = 20$. Thus, we will find acute angle B.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{20}{\sin B} = \frac{\sqrt{700}}{\sin 60^\circ}$$

$$\sqrt{700}\sin B = 20\sin 60^\circ$$

$$\sin B = \frac{20\sin 60^\circ}{\sqrt{700}} \approx 0.6547$$

$$B \approx 41^\circ$$

Step 3 **Find the third angle. Subtract the measure of the given angle and the angle found in step 2 from 180°.**

$$C = 180^\circ - A - B = 180^\circ - 60^\circ - 41^\circ = 79^\circ$$

The solution is $a = 26$, $B = 41^\circ$, and $C = 79^\circ$.

Solving an SSS Triangle

1. Use the Law of Cosines to find the angle opposite the longest side.
2. Use the Law of Sines to find either of the two remaining acute angles.
3. Find the third angle. Subtract the measures of the angles found in steps 1 and 2 from 180º.

Example 100

Solve the triangle ABC if a = 6, b = 9, c = 4.

Step 1: Use the Law of Cosines to find the angle opposite the longest side.

$b^2 = a^2 + c^2 - 2ac\cos B$ Solve for cosB

$cosB = \dfrac{a^2 + c^2 - b^2}{2ac}$ Enter your given side values

$\cos B = -\dfrac{29}{48}$ Since the cosine is negative, B is obtuse

$$B = \cos^{-1}\left(-\frac{29}{48}\right) \approx 127.2$$

Step 2: Apply the Law of Sines

Step 3: Find the third angle by subtraction.

Example 101 Applying Law of Cosines

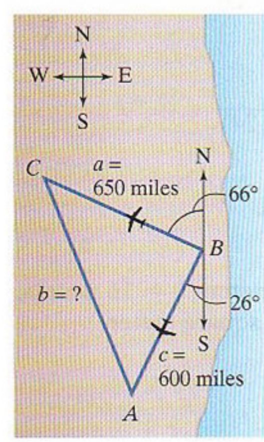
Two airplanes leave an airport at the same time on different runways. One flies at a bearing of N66ºW at 325 miles per hour. The other airplane flies at a bearing of S26ºW at 300 miles per hour. How far apart will the airplanes be after two hours?

Solution After two hours, the plane flying at 325 miles per hour travels $325 \cdot 2$ miles, or 650 miles. Similarly, the plane flying at 300 miles per hour travels 600 miles. The situation is illustrated in the figure.

Let b = the distance between the planes after two hours. We can use a north–south line to find angle B in triangle ABC. Thus,

$$B = 180° - 66° - 26° = 88°.$$

We now have $a = 650$, $c = 600$, and $B = 88°$.



Solution We use the Law of Cosines to find b in this SAS situation.

$b^2 = a^2 + c^2 - 2ac\cos B$ Apply the Law of Cosines.

$b^2 = 650^2 + 600^2 - 2(650)(600)\cos 88^\circ$ Substitute: a = 650, c = 600, and B = 88°.

After two hours, the planes are approximately 869 miles apart.

Heron's Formula

The area of a triangle with sides *a*, *b*, and *c* is

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}(a+b+c)$$

Example 102 Using Heron's Formula

Use Heron's formula to find the area of the given triangle:
a = 10m, b = 8m, c = 4m

$$s = \frac{1}{2}(a+b+c)$$

$$s = \frac{1}{2}(10+8+4)$$

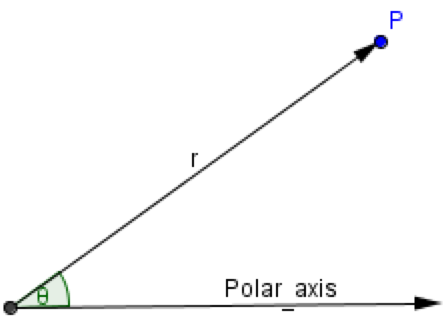
$$s = \frac{1}{2}(22) = 11$$

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{11(11-10)(11-8)(11-4)}$$

$$= \sqrt{11(1)(3)(7)} = \sqrt{231}\ sq.m.$$

Section 7.3 **Polar Coordinates**



The Sign of *r* and a Point's Location in Polar Coordinates:

The point $P = (r, \theta)$ is located $|r|$ units from the pole. If $r > 0$, the point lies on the terminal side of θ. If $r < 0$ the point lies along the ray opposite the terminal side of θ. If $r = 0$ the point lies at the pole, regardless of the value of θ.

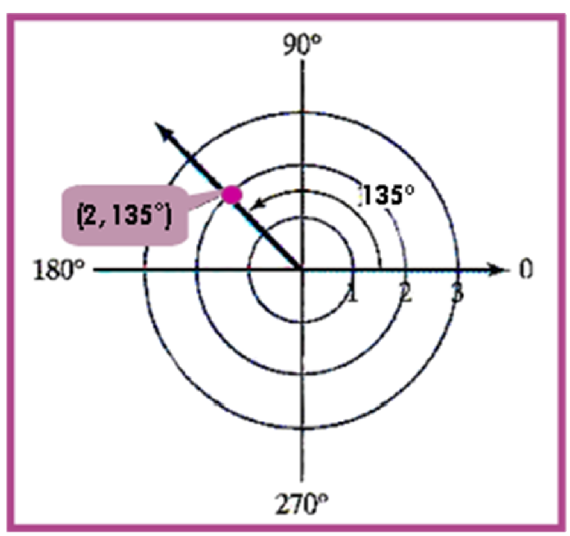
Example 103

Plot the points with the following polar coordinates:

a. (2, 135°)

Solution

a. To plot the point $(r, \theta) = (2, 135°)$, begin with the 135° angle. Because 135° is a positive angle, draw $\theta = 135°$ counterclockwise from the polar axis. Now consider $r = 2$. Because $r > 0$, plot the point by going out two units on the terminal side of θ. Figure (a) shows the point.



a. $\left(-3, \frac{3\pi}{2}\right)$ c. $\left(-1, -\frac{\pi}{4}\right)$

Multiple Representations of Points

In the rectangular coordinate system a point is uniquely represented by its x and y coordinates; however, this is not true for polar points. They have many representations:

If *n* is any integer, the point (*r*, *θ*) can be represented as

$(r, \theta) = (r, \theta + 2n\pi)$ or $(r, \theta) = (-r, \theta + \pi + 2n\pi)$

Example 104 Find another representation of $\left(5, \frac{\pi}{4}\right)$ in which:

a. $r > 0$ and $2\pi < \theta < 4\pi$
b. $r < 0$ and $0 < \theta < 2\pi$

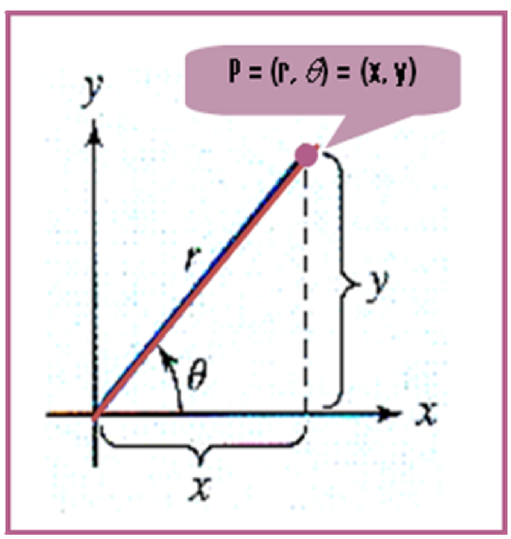
Relations between Polar and Rectangular Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$\tan\theta = y/x$$



Example 105 Find the rectangular coordinates for the following polar points:

a. $(3, \pi)$

b. $\left(-10, \frac{\pi}{6}\right)$

Converting a Point from Rectangular to Polar Coordinates
(r > 0 and $0 \leq \theta < 2\pi$)

1. Plot the point (x, y).
2. Find r by computing the distance from the origin to (x, y).
3. Find θ using tan θ= y/x with θ lying in the same quadrant as (x, y).

Example 106

Find the polar coordinates of a point whose rectangular coordinates are (2, 4)

Solution:

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$$

$$\tan\theta = \frac{y}{x} = \frac{4}{2} = 2$$

$$\theta = 1.1$$

$$(2\sqrt{5}, 1.1)$$

Example 107

Find the polar coordinates of a point whose rectangular coordinates are (0, -4)

Example 108 Converting an equation from Rectangular to Polar Coordinates

Convert 2x-y=1 to a polar equation.

Solution:

$$2x - y = 1$$

$$2r\cos\theta - r\sin\theta = 1$$

$$r(2\cos\theta - \sin\theta) = 1$$

$$r = \frac{1}{2\cos\theta - \sin\theta}$$

Converting Equations from Polar to Rectangular Form

Recall:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$\tan\theta = {}^{y}/x$$

We will use the above relationships to rewrite polar equations into rectangular form.

Example 109 Convert each polar equation to a rectangular equation in x and y:

a. $r = 4$

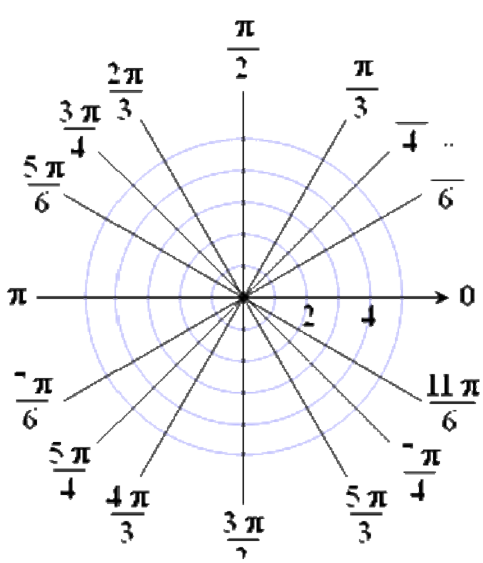
b. $\theta = \dfrac{3\pi}{4}$

c. $r = \sec\theta$

Section 7.4 **Graphs of Polar Equations**

Using Polar Grids to Graph Polar Equations

Recall that a **polar equation** is an equation whose variables are *r* and *è*. The **graph of a polar equation** is the set of all points whose polar coordinates satisfy the equation. We use **polar grids** like the one shown to graph polar equations. The grid consists of circles with centers at the pole. This polar grid shows five such circles. A polar grid also shows lines passing through the pole, In this grid, each fine represents an angle for which we know the exact values of the trigonometric functions.



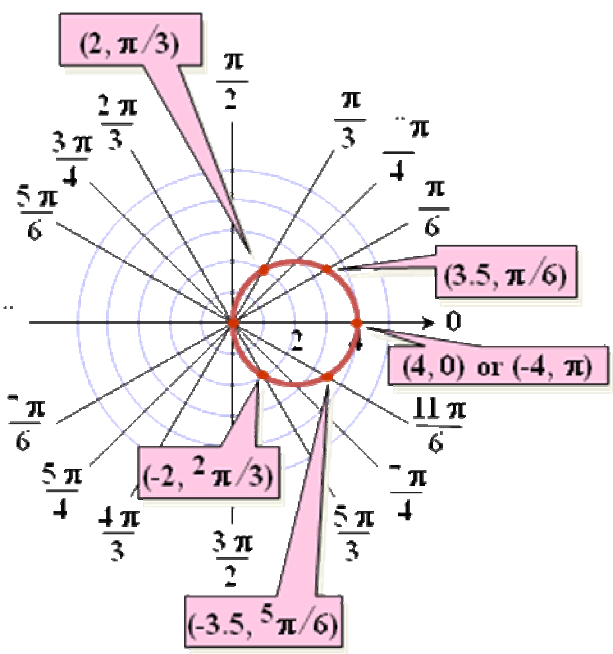
One method of graphing polar equations is to use point plotting. We will create a table of values just as we do with graphs in x and y.

Example 110

Graph the polar equation $r = 4\cos\theta$ with θ in radians.

Solution We construct a partial table of coordinates using multiples of $\pi/6$. Then we plot the points and join them with a smooth curve, as shown.

θ	$r = 4\cos\theta$	(r, θ)
0	$4\cos 0 = 4 \cdot 1 = 4$	(4, 0)
$\pi/6$	$4\cos\pi/6 = 4 \cdot \sqrt{3}/2 = 2\sqrt{3} = 3.5$	$(3.5, \pi/6)$
$\pi/3$	$4\cos\pi/3 = 4 \cdot 1/2 = 4$	$(2, \pi/3)$
$\pi/2$	$4\cos\pi/2 = 4 \cdot 0 = 0$	$(0, \pi/2)$
$2\pi/3$	$4\cos 2\pi/3 = 4(-1/2) = -2$	$(-2, 2\pi/3)$
$5\pi/6$	$4\cos 5\pi/6 = 4(-\sqrt{3}/2) = -2\sqrt{3} = -3.5$	$(-3.5, 5\pi/6)$
¨	$4\cos$ ¨ $\pi = 4(-1) = -4$	$(-4, \pi)$



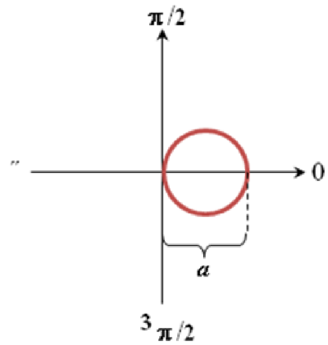
Circles in Polar Coordinates

The graphs of

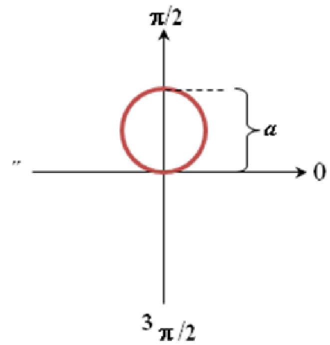
$r = a\cos\theta$ and $r = a\sin\theta$

Are circles.

$r = a\cos\theta$



$r = a\sin\theta$



Testing for Symmetry in Polar Coordinates (failure does not indicate a lack of symm.)

To test or symmetry with respect to the x-axis, replace θ with $-\theta$.

To test or symmetry with respect to the y-axis, replace (r,θ) with $(-r,-\theta)$.

To test or symmetry with respect to the origin, replace r with $-r$.

Example 111

Check for symmetry and then graph the polar equation: $r = 1 - \cos\theta$.

Solution We apply each of the tests for symmetry.

Polar Axis: Replace θ by $-\theta$ in $r = 1 - \cos\theta$:

$r = 1 - \cos(-\theta)$ **Replace θ by $-\theta$ in $r = 1 - \cos\theta$.**

$r = 1 - \cos\theta$ **The cosine function is even: $\cos(-\theta) = \cos\theta$.**

Because the polar equation does not change when θ is replaced by $-\theta$, the graph is symmetric with respect to the polar axis.

The Line $\theta = \pi/2$: Replace (r, θ) by $(-r, -\theta)$ in $r = 1 - \cos\theta$:

$-r = 1 - \cos(-\theta)$ **Replace r by $-r$ and θ by $-\theta$ in $-r = 1 - \cos(-\theta)$.**

$-r = 1 - \cos\theta$ **$\cos(-\theta) = \cos\theta$.**

$r = \cos\theta - 1$ **Multiply both sides by -1.**

Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when (r, θ) is replaced by $(-r, -\theta)$, the equation fails this symmetry test. The graph may of may not be symmetric with respect to the line $\theta = \pi/2$.

The Pole: Replace r by $-r$ in $r = 1 - \cos\theta$:

$-r = 1 - \cos\theta$ **Replace r by $-r$.**

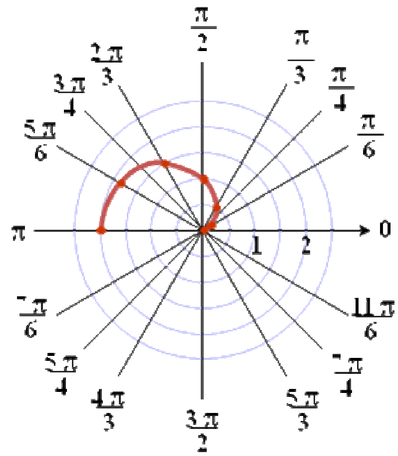
$r = \cos\theta - 1$ **Multiply both sides by -1.**

Because the polar equation $r = 1 - \cos\theta$ changes to $r = \cos\theta - 1$ when r is replaced by $-r$, the equation fails this symmetry test. The graph may or may not be symmetric with respect to the pole.

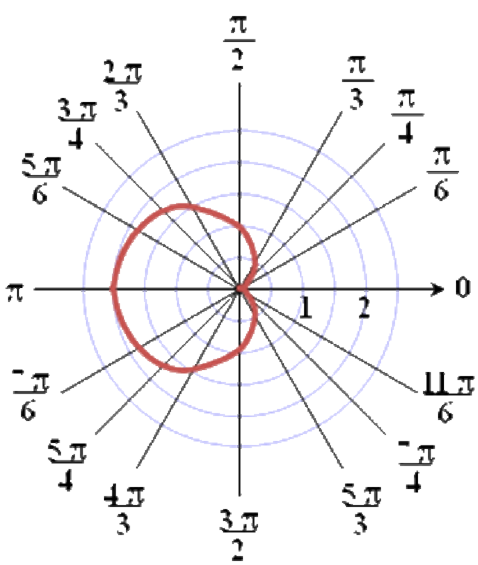
Now we are ready to graph $r = 1 - \cos\theta$. Because the period of the cosine function is $2r$, we need not consider values of θ beyond 2π. Recall that we discovered the graph of the equation $r = 1 - \cos\theta$ has symmetry with respect to the polar axis. Because the graph has symmetry, we may be able to obtain a complete graph without plotting points generated by values of θ from 0 to 2π. Let's start by finding the values of r for values of θ from 0 to π.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	0	0.13	0.50	1.00	1.50	1.87	2

The values for r and θ are in the table. Examine the graph. Keep in mind that the graph must be symmetric with respect to the polar axis.



Thus, if we reflect the graph from the last slide about the polar axis, we will obtain a complete graph of $r = 1 - \cos\ \theta$, shown below.



Example 112 Graph $r = 1 + 2\sin\theta$ (use symmetry to assist you)

Limacons

The graphs of

$$r = a + b\sin\theta, \quad r = a - b\sin\theta,$$
$$r = a + b\cos\theta, \quad r = a - b\cos\theta, \quad a > 0,\ b > 0$$

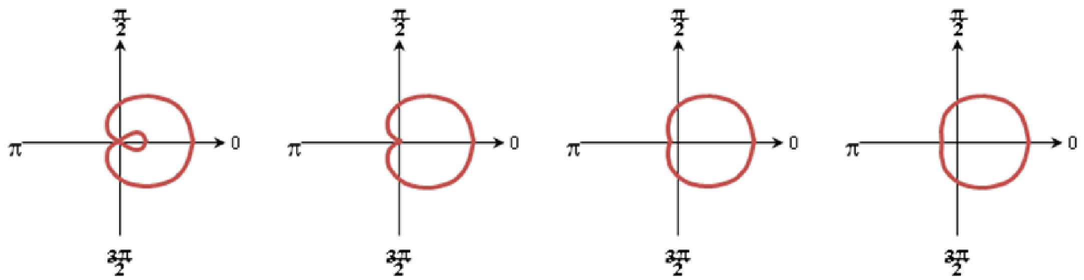
are called **limacons.** The ratio a/b determines a limacon's shape.

Inner loop if $a/b < 1$

Heart shaped if $a/b = 1$ and called cardiods

Dimpled with no inner loop if $1 < a/b < 2$

No dimple and no inner loop if $a/b \geq 2$.



Example 113

Graph the polar equation
y= 2+3cosθ

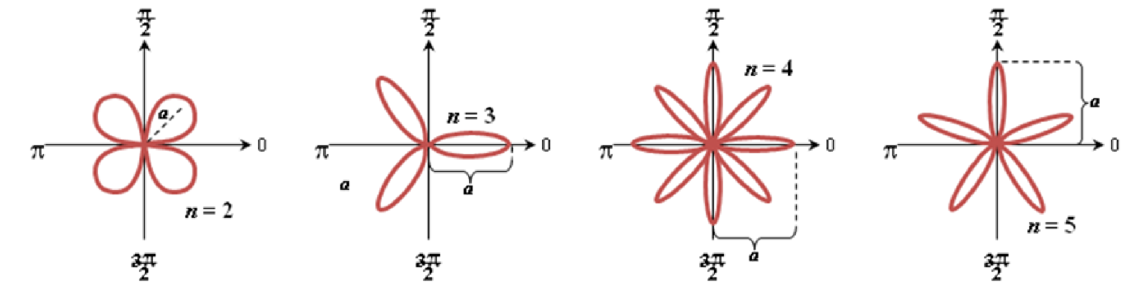
Rose Curves

The graphs of

$$r = a \sin n\theta \qquad \text{and} \qquad r = a \cos n\theta, \qquad a \textit{ does not equal } 0,$$

are called **rose curves**. If n is even, the rose has $2n$ petals. If n is odd, the rose has n petals.

$r = a \sin 2\theta$	$r = a \cos 3\theta$	$r = a \cos 4\theta$	$r = a \sin 5\theta$
Rose curve with 4 petals	Rose curve with 3 petals	Rose curve with 8 petals	Rose curve with 5 petals

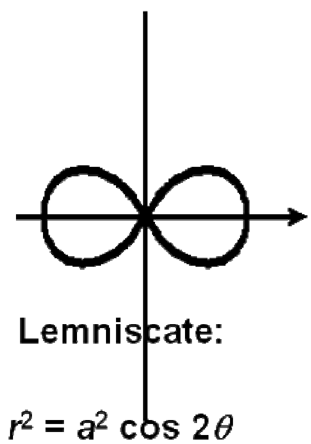


Example 114

Graph the polar equation y=3sin2θ

Lemniscates

- The graphs of $r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$ are called **lemniscates**



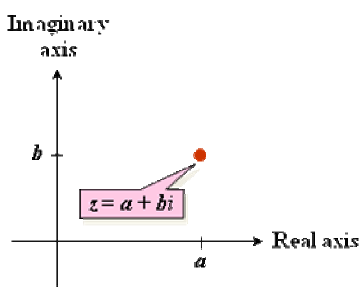
Example 115

Graph $r^2 = 4\sin 2\theta$

Section 7.5 Complex Numbers in Polar Form; DeMoivre's Theorem

The Complex Plane

We know that a real number can be represented as a point on a number line. By contrast, a complex number $z = a + bi$ is represented as a point (a, b) in a coordinate plane, shown below. The horizontal axis of the coordinate plane is called the **real axis**. The vertical axis is called the **imaginary axis**. The coordinate system is called the **complex plane**. Every complex number corresponds to a point in the complex plane and every point in the complex plane corresponds to a complex number.



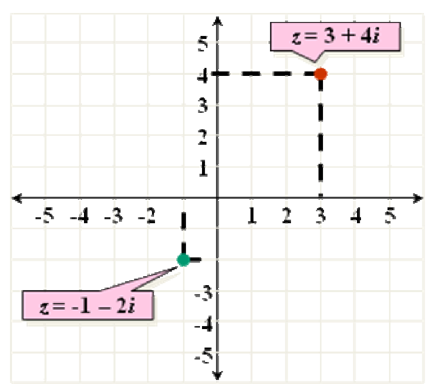
Example 116

Plot in the complex plane:

a. $z = 3 + 4i$ b. $z = -1 - 2i$ c. $z = -3$ d. $z = -4i$

Solution

- We plot the complex number $z = 3 + 4i$ the same way we plot $(3, 4)$ in the rectangular coordinate system. We move three units to the right on the real axis and four units up parallel to the imaginary axis.



- The complex number $z = -1 - 2i$ corresponds to the point $(-1, -2)$ in the rectangular coordinate system. Plot the complex number by moving one unit to the left on the real axis and two units down parallel to the imaginary axis.

The Absolute Value of a Complex Number

- The **absolute value** of the complex number a + bi is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

Example 117

Determine the absolute value of of each of the following complex numbers:

a. $z = 5 + 12i$

b. $z = 2 - 3i$

Example 118

Determine the absolute value of z=2-4i

$$\begin{aligned} |z| &= |a+bi| = \sqrt{a^2+b^2} \\ &= \sqrt{2^2+(-4)^2} = \sqrt{4+16} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Polar Form of a Complex Number

The complex number $a + bi$ is written in polar form as

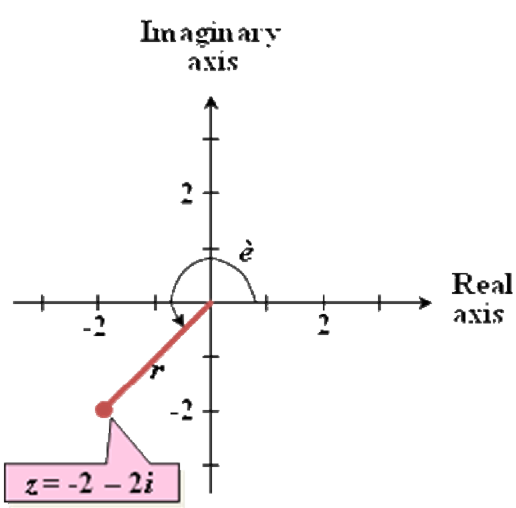
$$z = r\,(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$ The value of r is called the modulus (plural: moduli) of the complex number z, and the angle θ is called the argument of the complex number z, with $0 < \theta < 2\pi$

Example 119

Plot $z = -2 - 2i$ in the complex plane. Then write z in polar form.

Solution The complex number $z = -2 - 2i$, graphed below, is in rectangular form $a + bi$, with a = -2 and b = -2. By definition, the polar form of z is $r(\cos\theta + i\sin\theta)$. We need to determine the value for r and the value for θ, included in the figure below.



$$r = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{b}{a} = \frac{-2}{-2} = 1$$

$$z = r(\cos\theta + i\sin\theta) = 2\sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$$

Example 120 Writing a complex number in rectangular form:

Write $z = 4\left(\cos 30° + i\sin 30°\right)$ in rectangular form.

Solution: The complex number z is in polar form, with r = 4 and $\theta = 30°$. All we have to do is to evaluate the trigonometric functions in Z to get the rectangular form.

Thus $z = 4\left(\cos 30° + i\sin 30°\right) = 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = 2\sqrt{3} + 2i$

Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1 (\cos \theta_1 + \boldsymbol{i} \sin \theta_1)$ and

$z_2 = r_2 (\cos \theta_2 + \boldsymbol{i} \sin \theta_2)$ be two complex numbers in polar form. Their product, $z_1 z_2$, is

$$z_1 z_2 = r_1 r_2 (\cos (\theta_1 + \theta_2) + \boldsymbol{i} \sin (\theta_1 + \theta_2))$$

To multiply two complex numbers, multiply moduli and add arguments.

Example 121

Find the product of the complex numbers. Leave the answer in polar form.

$$z_1 = 4(\cos 50^\circ + i \sin 50^\circ) \qquad z_2 = 7(\cos 100^\circ + i \sin 100^\circ)$$

Solution

$z_1 z_2$

$= [4(\cos 50^\circ + i \sin 50^\circ)][7(\cos 100^\circ + i \sin 100^\circ)]$

$= (4 \cdot 7)[\cos (50^\circ + 100^\circ) + i \sin (50^\circ + 100^\circ)]$

$= 28(\cos 150^\circ + i \sin 150^\circ)$

Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ be two complex

numbers in polar form. Their quotient, z_1/z_2, is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

To divide two complex numbers, divide moduli and subtract arguments.

Example 122

Find the quotient of the complex numbers and leave your answer in polar form:

$z_1 = 50\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ and $z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

DeMoivre's Theorem

- Let $z = r(\cos\theta + i\sin\theta)$ be a complex numbers in polar form. If n is a positive integer, z to the nth power, z^n, is

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

Example 123

Find $[2(\cos 10° + i\sin 10°)]^6$. Write the answer in rectangular form a + bi.

Solution By DeMoivre's Theorem,

$[2(\cos 10° + i\sin 10°)]^6$

$= 2^6[\cos(6 \cdot 10°) + i\sin(6 \cdot 10°)]$	Raise the modulus to the 6th power and multiply the argument by 6.
$= 64(\cos 60° + i\sin 60°)$	Simplify.
$= 64\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$	Write the answer in rectangular form.
$= 32 + 32\sqrt{3}i$	Multiply and express the answer in $a + bi$ form.

Example 124

Find $(1+i)^8$ and write your answer in rectangular form.

DeMoivre's Theorem for Finding Complex Roots

- Let ω=r(cosθ+isinθ) be a complex number in polar form. If $\omega \neq 0$, ω has *n* distinct complex *n*th roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+360k}{n}\right)+i\sin\left(\frac{\theta+360k}{n}\right)\right]$$

$$where \quad k = 0,1,2,3,\ldots,n-1$$

Example 125

Find all the complex fourth roots of 81(cos60º+isin60º)

Solution:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+360k}{n}\right)+i\sin\left(\frac{\theta+360k}{n}\right)\right]$$

$$= \sqrt[4]{81}\left[\cos\left(\frac{60+360*0}{4}\right)+i\sin\left(\frac{60+360*0}{4}\right)\right]$$

$$= 3(\cos 15^{\circ} + i\sin 15^{\circ})$$

$$= \sqrt[4]{81}\left[\cos\left(\frac{60+360*1}{4}\right)+i\sin\left(\frac{60+360*1}{4}\right)\right]$$

$$= 3(\cos 105^{\circ} + i\sin 105^{\circ})$$

$$= 3(\cos 195^{\circ} + i\sin 195^{\circ})$$

$$= 3(\cos 285^{\circ} + i\sin 285^{\circ})$$

Example 126

Find all of the cube roots of 8 and express your answers in rectangular form.

Solution: Since DeMoivre's Theorem applies for the roots of complex numbers in polar form, we need to first write 8 into polar form.

$$8 = r(\cos\theta + i\sin\theta) = 8(\cos 0° + i\sin 0°)$$

$$z_0 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi*0}{3}\right) + i\sin\left(\frac{0+2\pi*0}{3}\right)\right]$$

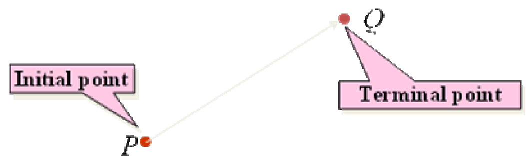
$$z_1 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi*1}{3}\right) + i\sin\left(\frac{0+2\pi*1}{3}\right)\right]$$

$$z_2 = \sqrt[3]{8}\left[\cos\left(\frac{0+2\pi*2}{3}\right) + i\sin\left(\frac{0+2\pi*2}{3}\right)\right]$$

Section 7.6 Vectors

Directed Line Segments and Geometric Vectors

A line segment to which a direction has been assigned is called a directed line segment. The figure below shows a directed line segment form P to Q. We call P the initial point and Q the terminal point. We denote this directed line segment by $\overrightarrow{PQ}$.



The magnitude of the directed line segment $\overrightarrow{PQ}$ is its length. We denote this by $\|\overrightarrow{PQ}\|$. Thus, $\|\overrightarrow{PQ}\|$ is the distance from point P to point Q. Because distance is nonnegative, vectors do not have negative magnitudes.

Geometrically, a vector is a directed line segment. Vectors are often denoted by a boldface letter, such as $\mathbf{v}$. If a vector $\mathbf{v}$ has the same magnitude and the same direction as the directed line segment $\overrightarrow{PQ}$, we write

$$\mathbf{v} = \overrightarrow{PQ}$$

The direction of a vector is determined by the slope of the line segment that connects the initial and terminal points of the directed line segment, so in order to find the direction of a vector, use the slope formula:

Direction of a vector = $\dfrac{(y_1 - y_2)}{(x_1 - x_2)}$

Example 127

Vector U has initial point (-3, -3) and terminal point (0, 3). Vector V has initial point (0, 0) and terminal point (3, 6). Show that vectors V and U are equal (i.e. -show they have the same magnitude and direction).

Component form of a Vector

Since for each vector there are an infinite number of equivalent vectors (vectors that have the same magnitude and direction), it is convenient to be able to use one vector to represent all of them. We will position this representative vector's initial point at the origin. We will call this ***placement standard position.***

Since every vector in standard position will have initial point (0, 0), vectors in standard position can be uniquely represented by their terminal point. This we will call the component form of the vector **v**.

Component form of vector **v** = $\langle v_1, v_2 \rangle$

***Note** the zero vector is denoted by **0** = $\langle 0,0 \rangle$

To write a vector into component form, simply subtract the x values of the terminal points and the initial points to get the x component of the vector, and then do the same for the y values:

Component Form of a Vector

The component form of a vector with initial point P = (p_1, p_2) and terminal point Q = (q_1, q_2) is given by $\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = v$

The magnitude then becomes: $\|v\| = \sqrt{v_1^2 + v_2^2}$

Note: if the magnitude of a vector is equal to one, it is said to be **a unit vector**.

Example 128 Find the component form and magnitude of the vector **v** that has initial point (4, -7) and terminal point (-1, 5).

Vector Operations:

Two common operations performed on vectors are scalar multiplication and vector addition.

<u>Addition</u>

Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$

The sum of **u + v** = $\langle u_1 + v_1, u_2 + v_2 \rangle$

<u>Scalar Multiplication</u>

Let k be a scalar (some real number)

Then k times **u** is the vector k**u** = $\langle ku_1, ku_2 \rangle$

Vector Multiplication

If k is a real number and **v** a vector, the vector k**v** is called a **scalar multiple** of the vector v. The magnitude and direction of k**v** are given as follows:

The vector k**v** has a magnitude of $|k|\ ||\mathbf{v}||$. We describe this as the absolute value of k times the magnitude of vector **v**.

The vector k**v** has a direction that is:

the same as the direction of **v** if $k > 0$, and

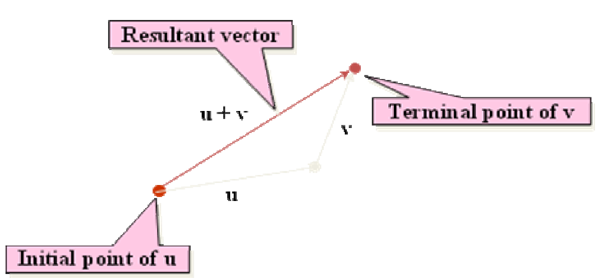
opposite the direction of **v** if $k < 0$

The Geometric Method for Adding Two Vectors

A geometric method for adding two vectors is shown below. The sum of **u** + **v** is called the **resultant vector**. Here is how we find this vector.

1. Position **u** and **v** so the terminal point of **u** extends from the initial point of **v**.

2. The resultant vector, **u** + **v**, extends from the initial point of **u** to the terminal point of **v**.



Example 129 Let $v = \langle -2, 5 \rangle$ and $w = \langle 3, 4 \rangle$, and find each of the following vectors.

a. 2v   b. w – v   c. v + 2w   d. 2v – 3w

Properties of Vector Addition and Scalar Multiplication

1. u + v = v + u
2. (u + v) + w = u + (v + w)
3. u + 0 = u
4. u + (-u) = 0
5. c(du) = (cd)u
6. (c + d)u = cu + du
7. c(u + v) = cu + cv
8. 1(u) = u, 0(u) = **0**
9. $\|cv\| = |c|\|v\|$

Unit Vectors

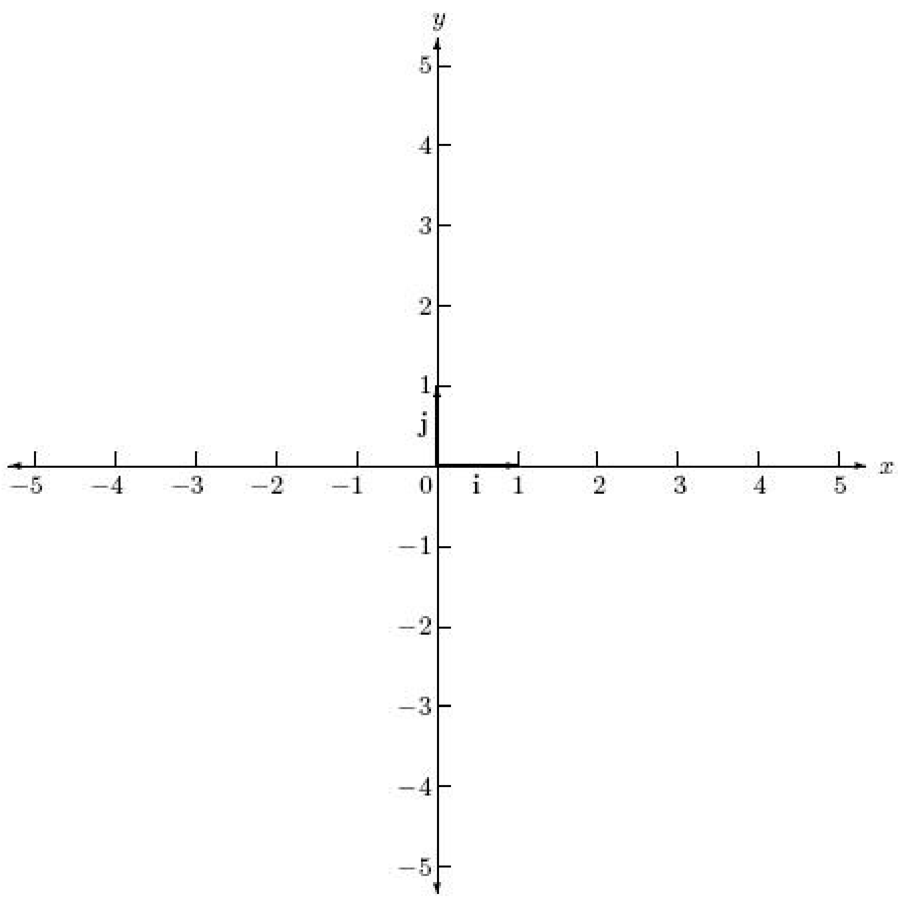
In many applications of vectors it is useful to find something called a unit vector that has the same direction as some given vector. Recall that a unit vector is just a vector that has a magnitude of one. To find a unit vector in the same direction as some other vector, **v**, we simply divide **v** by its magnitude (think scalar multiplication by 1/$\|v\|$) .

Unit vector in the direction of v = $\dfrac{v}{\|v\|} = \dfrac{1}{\|v\|}\langle v_1, v_2 \rangle$

Example 130 Find a unit vector in the direction of v = < -2, 5 > and verify it has a magnitude of 1.

The unit vectors <1 , 0> and <0, 1> are called the standard unit vectors and are denoted by:

$i = \langle 1,0 \rangle$ and $j = \langle 0,1 \rangle$



Any vector can be written as a **linear combination** of the **i** and **j** vectors, for example:

$$v = \langle v_1, v_2 \rangle = v_1 \langle 1,0 \rangle + v_2 \langle 0,1 \rangle = v_1 i + v_2 j$$

The scalars v_1 *and* v_2 are called the horizontal and vertical components of **v** respectively.

***Note: a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results together.

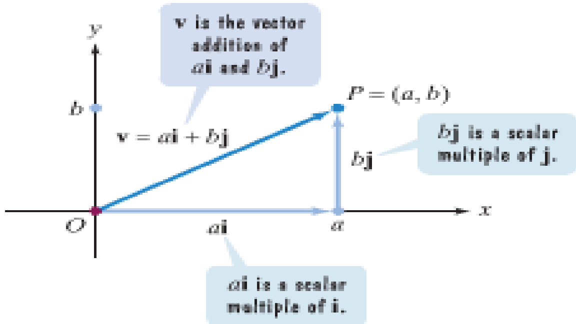


Figure 6.55 Using vector addition, vector $\mathbf{v}$ is represented as $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

Representing Vectors in Rectangular Coordinates

Vector $\mathbf{v}$, from $(0, 0)$ to (a, b), is represented as

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j}.$$

The real numbers a and b are called the **scalar components of v**. Note that

- a is the **horizontal component** of $\mathbf{v}$, and
- b is the **vertical component** of $\mathbf{v}$.

The vector sum $a\mathbf{i} + b\mathbf{j}$ is called a **linear combination** of the vectors $\mathbf{i}$ and $\mathbf{j}$.
The magnitude of $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is given by

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}.$$

Example 131 Let u be the vector with initial point (5, 9) and terminal point (-1, 4), write u as a linear combination of the **i** and **j** vectors.

Adding and Subtracting Vectors in Terms of i and j

If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$, then

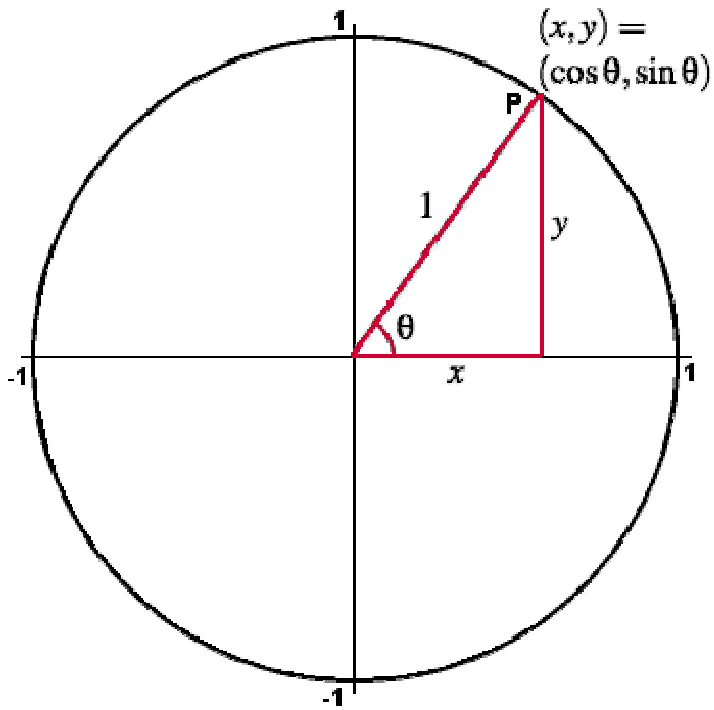
$$\mathbf{v} + \mathbf{w} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$$
$$\mathbf{v} - \mathbf{w} = (a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}.$$

Example 132 Let $u = -3i + 7j$ and $v = 3i - 8j$ then Find 2u - 4v

Direction Angles

If $u = \langle x, y \rangle$ is a unit vector such that θ is the angle measured from the positive x –axis to **u, the terminal point of u lies on the unit circle and you have:**

$$u = \langle x, y \rangle = \langle \cos\theta, \sin\theta \rangle = (\cos\theta)i + (\sin\theta)\,j$$



If $v = ai + bj$ is any vector with direction angle θ measured from the positive x –axis, we can write:

$$v = ai + bj = \|v\|\langle \cos\theta, \sin\theta \rangle = \|v\|(\cos\theta)i + \|v\|(\sin\theta)\,j$$

Since $v = ai + bj = \|v\|\langle \cos\theta, \sin\theta \rangle = \|v\|(\cos\theta)i + \|v\|(\sin\theta)\,j$, we can see that the direction angle θ for **v** can be found by using the expression : $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\|v\|(\sin\theta)}{\|v\|(\cos\theta)} = \dfrac{b}{a}$

Thus by **using inverse tangent** we get theta: $\tan^{-1}(b/a) = \theta$

Example 133 Find the direction angle for each of the vectors: **v** = 2i + 2j and **w** = 3i – 4j

Writing a Vector in Terms of Its Magnitude and Direction

Let $\mathbf{v}$ be a nonzero vector. If θ is the direction angle measured from the positive x-axis to $\mathbf{v}$, then the vector can be expressed in terms of its magnitude and direction angle as

$$\mathbf{v} = \|\mathbf{v}\| \cos \theta \mathbf{i} + \|\mathbf{v}\| \sin \theta \mathbf{j}.$$

Applications

Example 134 Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle of 30 degrees below the horizontal.

Solution: 100(cos210)i + 100(sin2109)j = $\left\langle -50\sqrt{3}, -50 \right\rangle$