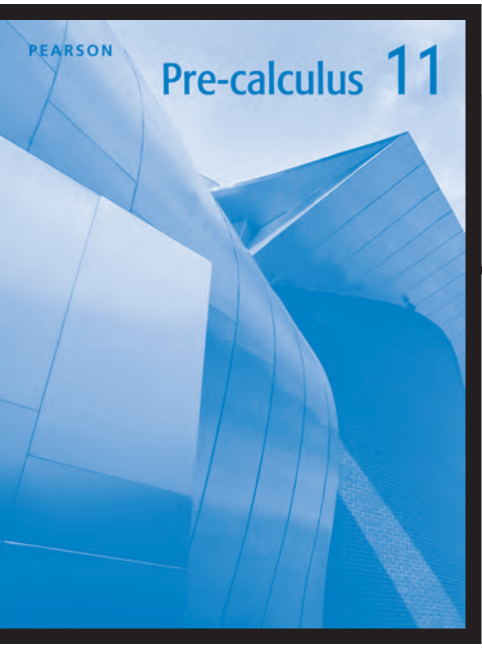
PEARSON

Pre-calculus 11

TEACHER RESOURCE



1 Sequences and Series

Planning for the Chapter

Content Pages with Teaching Notes

Line Masters

CHAPTER OVERVIEW

Background

In grade 9, students graphed linear relations and solved linear equations, which is necessary background knowledge for understanding arithmetic sequences and series. In grade 10, these concepts are extended to include linear functions. By the time students enter grade 11, they should be able to determine the slope of the graph of a linear function. This is related to the constant difference in an arithmetic sequence.

In grade 9, students were introduced to powers and roots (limited to whole number exponents) and square roots of perfect squares. In grade 10, these concepts were extended to include integral and rational exponents and to determining nth roots. This knowledge is a prerequisite for solving problems involving geometric sequences and series.

Rationale

Sequences and series are introduced in grade 11. Students use their understanding of linear functions to develop the properties of arithmetic sequences and series, then solve related problems. They derive rules for determining the nth term of an arithmetic sequence and the sum of the first n terms of an arithmetic series.

Students are introduced to geometric sequences and series, and distinguish them from arithmetic sequences and series. They derive rules for determining the nth term of a geometric sequence and the sum of the first n terms of a geometric series. Students solve problems that can be modelled using geometric sequences and series.

The concept of convergence and divergence of infinite geometric sequences and series is introduced through graphing. Students see that the points on some graphs approach a horizontal line while the points on other graphs move away from a horizontal line. They develop an informal understanding about the role of the common ratio in determining convergence and divergence, and derive a rule to determine whether an infinite geometric sequence or series converges. Students learn that the terms of a convergent geometric sequence approach 0 as the term number increases, and they derive a rule to determine the sum of an infinite geometric series that converges.

Concept Summary

Big Ideas	Applying the Big Ideas
• An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.	This means that: • The common difference of an arithmetic sequence is equal to the slope of the line through the points of the graph of the related linear function. • Rules can be derived to determine the *n*th term of an arithmetic sequence and the sum of the first *n* terms of an arithmetic series.
• A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.	• The common ratio of a geometric sequence can be determined by dividing any term after the first term by the preceding term. • Rules can be derived to determine the *n*th term of a geometric sequence and the sum of the first *n* terms of a geometric series.
• Any finite series has a sum, but an infinite geometric series may or may not have a sum.	• The common ratio determines whether an infinite series has a finite sum.

CURRICULUM OVERVIEW

Grade 9	Foundations of Mathematics and Pre-calculus 10	Pre-calculus 11	Pre-calculus 12
		Chapter 1 Outcomes	
PR2 Graph linear relations, analyze the graph and interpolate or extrapolate to solve problems. **PR3** Model and solve problems using linear equations. **N1** Demonstrate an understanding of powers with integral bases (excluding base 0) and whole number exponents by: • representing repeated multiplication using powers • using patterns to show that a power with an exponent of zero is equal to one • solving problems involving powers.	**RF3** Demonstrate an understanding of slope with respect to: rise and run; line segments and lines; rate of change; parallel lines; perpendicular lines. **RF4** Describe and represent linear relations, using: words; ordered pairs; tables of values; graphs; equations. **RF5** Determine the characteristics of the graphs of linear relations, including the: intercepts; slope; domain; range. **AN3** Demonstrate an understanding of powers with integral and rational exponents.	**RF9** Analyze arithmetic sequences and series to solve problems. [CN, PS, R, T] **RF10** Analyze geometric sequences and series to solve problems. [PS, R, T]	**RF9** Graph and analyze exponential and logarithmic functions. **RF10** Solve problems that involve exponential and logarithmic equations.

CHAPTER 1 AT A GLANCE

Lesson	Timing	Materials and Resources	Program Support
Chapter Opener, page 1 Review prior skills, as relevant.			Masters 1.1a, 1.1b Activate Prior Learning Master 1.6 Chapter Rubric
1.1 Arithmetic Sequences, page 2 Relate linear functions and arithmetic sequences, then solve problems related to arithmetic sequences.	60 – 75 min	• grid paper • scientific calculator • graphing calculator (*optional*) Dynamic Activity; Animation	Master 1.1a Activate Prior Learning
1.2 Arithmetic Series, page 14 Derive a rule to determine the sum of *n* terms of an arithmetic series, then solve related problems.	75 min	• scientific calculator • graphing calculator (*optional*) Animations	PM 1 Mathematical Dispositions and Learning Skills
Checkpoint 1, page 25 Consolidate content of Lessons 1.1, 1.2.		• scientific calculator	Master 1.3 Checkpoint 1
1.3 Geometric Sequences, page 29 Solve problems involving geometric sequences.	75 min	• scientific calculator • graphing calculator (*optional*) Animation	Master 1.1b Activate Prior Learning
1.4 Geometric Series, page 43 Derive a rule to determine the sum of *n* terms of a geometric series, then solve related problems.	75 min	• scientific calculator • graphing calculator (*optional*) Animations	
Checkpoint 2, page 55 Consolidate content of Lessons 1.3, 1.4.		• scientific calculator	Master 1.4 Checkpoint 2 PM 2 Conference Prompts
1.5 Math Lab: Graphing Geometric Sequences and Series, page 58 Investigate the graphs of geometric sequences and geometric series.	60 – 75 min	• graphing calculator, or computer with graphing software • grid paper (*optional*) Dynamic Activity	Master 1.2 Math Lab: Graphing Calculator Instructions PM 3 Observation Record
1.6 Infinite Geometric Series, page 63 Determine the sum of an infinite geometric series.	75 min	• scientific calculator • graphing calculator (*optional*)	
Study Guide, Review, Practice Test, page 74 Consolidate and review chapter content, prepare for assessment.		• scientific calculator	Master 1.5 Chapter Test Master 1.7 Chapter Rubric Master 1.8 Chapter Summary

The DVD provides:
All Program Masters and Chapter 1 masters, as editable and pdf files
SMART Notebook files for all lessons
Extra Material for selected lessons

1 Sequences and Series

BUILDING ON

- graphing linear functions
- properties of linear functions
- expressing powers using exponents
- solving equations

BIG IDEAS

- An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.
- A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.
- Any finite series has a sum, but an infinite geometric series may or may not have a sum.

LEADING TO

- applying the properties of geometric sequences and series to functions that illustrate growth and decay

NEW VOCABULARY

arithmetic sequence	common ratio
term of a sequence or series	finite and infinite geometric sequences
common difference	divergent and convergent sequences
infinite arithmetic sequence	geometric series
general term	infinite geometric series
series	sum to infinity
arithmetic series	
geometric sequence	

TEACHER NOTE

DI: Prerequisite Review

For those students who need it, Master 1.1 provides review (examples and selected exercises) of prior knowledge related to:

a) slopes of the graphs of linear functions;

b) powers and roots

Master	Relevant for:
1.1a	Lesson 1.1
1.1b	Lesson 1.3

1.1 Arithmetic Sequences

Lesson Organizer
60 – 75 min

Key Math Concepts
The difference between consecutive terms in an arithmetic sequence is constant. We can use the patterns in the terms to derive a rule for determining the *n*th term.

Curriculum Focus

SO	AI
RF9	9.1, 9.2, 9.3, 9.4, 9.5, 9.8

Processes: CN, PS, R, T

Teacher Materials
- overhead transparency of grid paper (*optional*)

Student Materials
- grid paper
- scientific calculator
- graphing calculator (*optional*)
- Master 1.1a (*optional*)

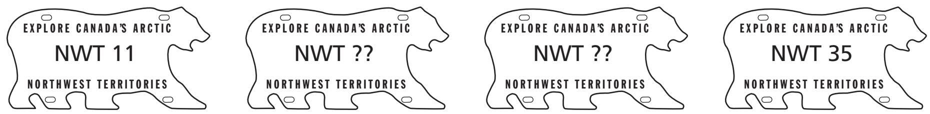
Vocabulary
arithmetic sequence, term, common difference, infinite arithmetic sequence, general term

Extra Material
Definitions of a sequence and a term; Fibonacci sequence

FOCUS **Relate linear functions and arithmetic sequences, then solve problems related to arithmetic sequences.**

Get Started

When the numbers on these plates are arranged in order, the differences between each number and the previous number are the same.



What are the missing numbers?

Let the difference between each pair of numbers be x.

Then, $11 + 3x = 35$
$$3x = 24$$
$$x = 8$$
The numbers are 19 and 27.

Construct Understanding

Saket took guitar lessons.

The first lesson cost \$75 and included the guitar rental for the period of the lessons.

The total cost for 10 lessons was \$300.

Suppose the lessons continued.

What would be the total cost of 15 lessons?

Find the cost of one extra lesson.

Cost of 9 lessons = total cost of 10 lessons − total cost of 1st lesson
= \$300 − \$75
= \$225

So, the cost of 1 lesson is: $\frac{\$225}{9} = \$25$

The cost of 15 lessons is: 1 lesson @ \$75 + 14 lessons @ \$25
So, 15 lessons cost: \$75 + 14(\$25) = \$425

In an **arithmetic sequence**, the difference between consecutive **terms** is constant. This constant value is called the **common difference**.

This is an arithmetic sequence:
4, 7, 10, 13, 16, 19, . . .
The first term of this sequence is: $t_1 = 4$
The second term is: $t_2 = 7$

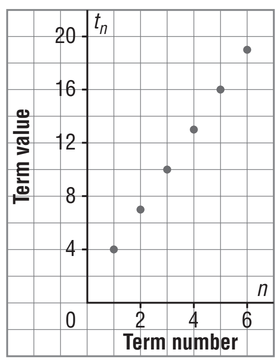
Let d represent the common difference. For the sequence above:

$$\begin{aligned} d &= t_2 - t_1 \\ &= 7 - 4 \\ &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} d &= t_3 - t_2 \\ &= 10 - 7 \\ &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} d &= t_4 - t_3 \\ &= 13 - 10 \\ &= 3 \end{aligned} \quad \text{and so on}$$

$$4, \overset{+3}{\frown} 7, \overset{+3}{\frown} 10, \overset{+3}{\frown} 13, \overset{+3}{\frown} 16, \overset{+3}{\frown} 19, \ldots$$

The dots indicate that the sequence continues forever; it is an **infinite arithmetic sequence**.
To graph this arithmetic sequence, plot the term value, t_n, against the term number, n.

Graph of an Arithmetic Sequence



The graph represents a linear function because the points lie on a straight line. A line through the points on the graph has slope 3, which is the common difference of the sequence.

In an arithmetic sequence, the common difference can be any real number.

Here are some other examples of arithmetic sequences.

- This is an *increasing* arithmetic sequence because d is positive and the terms are increasing:
  $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, \ldots$; with $d = \frac{1}{4}$
- This is a *decreasing* arithmetic sequence because d is negative and the terms are decreasing:
  $5, -1, -7, -13, -19, \ldots$; with $d = -6$

TEACHER NOTE

DI: Extending Thinking
Student page 2: Have students investigate how varying the cost of the first lesson, which includes the guitar rental, affects the rule for determining the cost of n lessons. For example, if the first lesson cost \$120, the cost of each lesson would be \$20 and the guitar rental would be \$100. The cost in dollars for n lessons would be $100 + 20n$.

THINK FURTHER

Why is the domain of every arithmetic sequence the natural numbers?

Because the natural numbers label the positions of the terms.

Dynamic Activity

Extra Material

Graphs of increasing and decreasing sequences

THINK FURTHER

What sequence is created when the common difference is 0?

The sequence is the first term repeated.

Example 1 Writing an Arithmetic Sequence

Write the first 5 terms of:

a) an increasing arithmetic sequence

b) a decreasing arithmetic sequence

SOLUTION

a) Choose any number as the first term; for example, $t_1 = -7$.
The sequence is to increase, so choose a positive common difference; for example, $d = 2$. Keep adding the common difference until there are 5 terms.

$$\begin{array}{ccccccccc} & +2 & & +2 & & +2 & & +2 & \\ -7, & & -5, & & -3, & & -1, & & 1, \dots \\ t_1 & & t_2 & & t_3 & & t_4 & & t_5 \end{array}$$

The arithmetic sequence is: $-7, -5, -3, -1, 1, \dots$

b) Choose the first term; for example, $t_1 = 5$.
The sequence is to decrease, so choose a negative common difference; for example, $d = -3$.

$$\begin{array}{ccccccccc} & -3 & & -3 & & -3 & & -3 & \\ 5, & & 2, & & -1, & & -4, & & -7, \dots \\ t_1 & & t_2 & & t_3 & & t_4 & & t_5 \end{array}$$

The arithmetic sequence is: $5, 2, -1, -4, -7, \dots$

Check Your Understanding

1. Write the first 6 terms of:

a) an increasing arithmetic sequence

b) a decreasing arithmetic sequence

a) Sample response: Choose $t_1 = -20$ and $d = 2$.
The sequence is: $-20, -20 + 2, -20 + 4, -20 + 6, -20 + 8, -20 + 10, \dots$
Simplify. An arithmetic sequence is: $-20, -18, -16, -14, -12, -10, \dots$

b) Sample response: Choose $t_1 = 100$ and $d = -3$.
The sequence is: $100, 100 - 3, 100 - 6, 100 - 9, 100 - 12, 100 - 15, \dots$
Simplify. An arithmetic sequence is: $100, 97, 94, 91, 88, 85, \dots$

Consider this arithmetic sequence: $3, 7, 11, 15, 19, 23, \dots$
To determine an expression for the **general term**, t_n, use the pattern in the terms. The common difference is 4. The first term is 3.

t_1	$3 = 3 + 4(0)$
t_2	$7 = 3 + 4(1)$
t_3	$11 = 3 + 4(2)$
t_4	$15 = 3 + 4(3)$
⋮	
t_n	$3 + 4(n - 1)$

For each term, the second factor in the product is 1 less than the term number.

The second factor in the product is 1 less than n, or $n - 1$.

Write: $\underset{\text{general term}}{t_n} = \underset{\text{first term}}{3} + \underset{\text{common difference}}{4}(n - 1)$

Check Your Understanding

Answers:

1. a) $-20, -18, -16, -14, -12, -10, \dots$

b) $100, 97, 94, 91, 88, 85, \dots$

The General Term of an Arithmetic Sequence

An arithmetic sequence with first term, t_1, and common difference, d, is: $t_1, t_1 + d, t_1 + 2d, t_1 + 3d, \ldots$

The general term of this sequence is: $t_n = t_1 + d(n - 1)$

TECHNOLOGY NOTE

Graphing Calculator

One way to display the terms of an arithmetic sequence: use the general term of the sequence to define a function in sequence mode, then use the table feature to view its terms.

Example 2 Calculating Terms in a Given Arithmetic Sequence

For this arithmetic sequence: $-3, 2, 7, 12, \ldots$

a) Determine t_{20}.

b) Which term in the sequence has the value 212?

SOLUTION

$-3, 2, 7, 12, \ldots$

a) Calculate the common difference: $2 - (-3) = 5$

Use: $t_n = t_1 + d(n - 1)$ Substitute: $n = 20, t_1 = -3, d = 5$

$t_{20} = -3 + 5(20 - 1)$ Use the order of operations.

$t_{20} = -3 + 5(19)$

$t_{20} = 92$

b) Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_n = 212, t_1 = -3, d = 5$

$212 = -3 + 5(n - 1)$ Solve for n.

$212 = -3 + 5n - 5$

$220 = 5n$

$\frac{220}{5} = n$

$n = 44$

The term with value 212 is t_{44}.

Check Your Understanding

2. For this arithmetic sequence: 3, 10, 17, 24, . . .

a) Determine t_{15}.

b) Which term in the sequence has the value 220?

a) **The common difference is:**
$10 - 3 = 7$
Use: $t_n = t_1 + d(n - 1)$
Substitute: $n = 15, t_1 = 3, d = 7$
$t_{15} = 3 + 7(15 - 1)$
$t_{15} = 101$

b) **Use: $t_n = t_1 + d(n - 1)$**
Substitute: $t_n = 220, t_1 = 3, d = 7$
$220 = 3 + 7(n - 1)$
$224 = 7n$
$n = 32$

The term with value 220 is t_{32}.

THINK FURTHER

In *Example 2*, how could you show that 246 is *not* a term of the sequence?

I would substitute $t_n = 246, t_1 = -3$, and $d = 5$ in $t_n = t_1 + d(n - 1)$, and when I solved for n, its value would not be a natural number.

Check Your Understanding

Answers:

2. a) 101

b) t_{32}

Animation

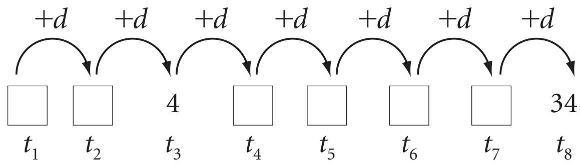
Example 3 Calculating a Term in an Arithmetic Sequence, Given Two Terms

Two terms in an arithmetic sequence are $t_3 = 4$ and $t_8 = 34$. What is t_1?

SOLUTION

$t_3 = 4$ and $t_8 = 34$

Sketch a diagram. Let the common difference be d.



From the diagram,

$t_8 = t_3 + 5d$ Substitute: $t_8 = 34$, $t_3 = 4$

$34 = 4 + 5d$ Solve for d.

$30 = 5d$

$d = 6$

Then, $t_1 = t_3 - 2d$ Substitute: $t_3 = 4$, $d = 6$

$t_1 = 4 - 2(6)$

$t_1 = 4 - 12$

$t_1 = -8$

Check Your Understanding

3. Two terms in an arithmetic sequence are $t_4 = -4$ and $t_7 = 23$. What is t_1?

There are 3 terms from t_4 to t_7.
So, $t_7 = t_4 + 3d$
Substitute: $t_7 = 23$, $t_4 = -4$
$23 = -4 + 3d$
Solve for d.
$3d = 27$
$d = 9$
$t_1 = t_4 - 3d$
$t_1 = -4 - 3(9)$
$t_1 = -31$

Extra Material

Alternative solution method for *Example 3*

Example 4 Using an Arithmetic Sequence to Model and Solve a Problem

Some comets are called periodic comets because they appear regularly in our solar system. The comet Kojima appears about every 7 years and was last seen in the year 2007. Halley's comet appears about every 76 years and was last seen in 1986.
Determine whether both comets should appear in 3043.

SOLUTION

The years in which each comet appears form an arithmetic sequence.
The arithmetic sequence for Kojima has $t_1 = 2007$ and $d = 7$.
To determine whether Kojima should appear in 3043, determine whether 3043 is a term of its sequence.

$t_n = t_1 + d(n - 1)$ Substitute: $t_n = 3043$, $t_1 = 2007$, $d = 7$

$3043 = 2007 + 7(n - 1)$ Solve for n.

$3043 = 2000 + 7n$

$1043 = 7n$

$149 = n$

Check Your Understanding

4. The comet Denning-Fujikawa appears about every 9 years and was last seen in the year 2005.

Determine whether the comet should appear in 3085.

The years the comet appears form an arithmetic sequence with:
$t_1 = 2005$ and $d = 9$
Use: $t_n = t_1 + d(n - 1)$ ▶

Check Your Understanding

Answers:

3. –31

4. The comet should appear in 3085.

Since the year 3043 is the 149th term in the sequence, Kojima should appear in 3043.

The arithmetic sequence for Halley's comet has $t_1 = 1986$ and $d = 76$. To determine whether Halley's comet should appear in 3043, determine whether 3043 is a term of its sequence.

$t_n = t_1 + d(n - 1)$ Substitute: $t_n = 3043$, $t_1 = 1986$, $d = 76$
$3043 = 1986 + 76(n - 1)$ Solve for n.
$3043 = 1910 + 76n$
$1133 = 76n$
$n = 14.9078\ldots$

Since n is not a natural number, the year 3043 is not a term in the arithmetic sequence for Halley's comet; so the comet will not appear in that year.

Substitute: $t_n = 3085$, $t_1 = 2005$, $d = 9$
$3085 = 2005 + 9(n - 1)$
$3085 = 1996 + 9n$
$1089 = 9n$
$121 = n$

Since the year 3085 is the 121st term in the sequence, the comet should appear in 3085.

Discuss the Ideas

1. How can you tell whether a sequence is an arithmetic sequence? What do you need to know to be certain?

I would calculate the difference between the given consecutive terms. If the differences are equal, the sequence is possibly arithmetic. I can't be certain unless I know that the sequence continues with the same difference between terms.

2. The definition of an arithmetic sequence relates any term after the first term to the preceding term. Why is it useful to have a rule for determining any term?

When I use a rule, I don't have to write all the terms before the term I'm trying to determine. For example, to determine t_{50}, if I didn't have a rule, I would have to write all the terms from t_1 to t_{50}.

3. Suppose you know a term of an arithmetic sequence. What information do you need to determine any other term?

I need to know the position, n, of the given term, t_n, and the common difference, d. Then I can substitute in the expression for t_n to determine t_1. Once I know t_1 and d, I can determine the value of any other term.

TEACHER NOTE

DI: Common Difficulties

For students who have difficulty deriving the rule for the nth term of an arithmetic sequence, suggest they compare the sequence to multiples of the common difference. For example:

t_1	t_2	t_3	t_4	
4	7	10	13	$d = 3$

Look at multiples of 3 (think of skip counting):
3 6 9 12

Each term in the sequence is 1 more than the corresponding multiple of 3. So, the rule for the nth term is:
$t_n = 3n + 1$, which is equivalent to:
$t_n = 4 + 3(n - 1)$

Exercises

A

4. Circle each sequence that could be arithmetic. Determine its common difference, d.

a) 6, 10, 14, 18, . . .

d is: $10 - 6 = 4$

b) 9, 7, 5, 3, . . .

d is: $7 - 9 = -2$

c) −11, −4, 3, 10, . . .

d is: $-4 - (-11) = 7$

d) 2, −4, 8, −16, . . .

Not arithmetic

5. Each sequence is arithmetic. Determine each common difference, d, then list the next 3 terms.

a) 12, 15, 18, . . .

d is: $15 - 12 = 3$
The next 3 terms are:
$18 + 3, 18 + 6, 18 + 9$;
or 21, 24, 27

b) 25, 21, 17, . . .

d is: $21 - 25 = -4$
The next 3 terms are:
$17 - 4, 17 - 8, 17 - 12$;
or 13, 9, 5

6. Determine the indicated term of each arithmetic sequence.

a) 6, 11, 16, . . . ; t_7

Use: $t_n = t_1 + d(n - 1)$
Substitute: $n = 7, t_1 = 6, d = 5$
$t_7 = 6 + 5(7 - 1)$
$t_7 = 36$

b) 2, $1\frac{1}{2}$, 1, . . . ; t_{35}

Use: $t_n = t_1 + d(n - 1)$
Substitute: $n = 35, t_1 = 2, d = -\frac{1}{2}$
$t_{35} = 2 - \frac{1}{2}(35 - 1)$
$t_{35} = -15$

7. Write the first 4 terms of each arithmetic sequence, given the first term and the common difference.

a) $t_1 = -3, d = 4$

$t_1 = -3$
t_2 is $t_1 + d = -3 + 4$, or 1
t_3 is $t_2 + d = 1 + 4$, or 5
t_4 is $t_3 + d = 5 + 4$, or 9

b) $t_1 = -0.5, d = -1.5$

$t_1 = -0.5$
t_2 is $t_1 + d = -0.5 - 1.5$, or -2
t_3 is $t_2 + d = -2 - 1.5$, or -3.5
t_4 is $t_3 + d = -3.5 - 1.5$, or -5

B

8. When you know the first term and the common difference of an arithmetic sequence, how can you tell if it is increasing or decreasing? Use examples to explain.

An arithmetic sequence is increasing if d is positive; for example, when t_1 is -10 and $d = 3$: −10, −7, −4, −1, 2, . . .
An arithmetic sequence is decreasing if d is negative; for example, when t_1 is -10 and $d = -3$: −10, −13, −16, −19, −22, . . .

9. **a)** Create your own arithmetic sequence. Write the first 7 terms. Explain your method.

Sample response: I chose $t_1 = 3$ and $d = 5$; I add 5 to 3, then keep adding 5. The first 7 terms of the sequence are: 3, 8, 13, 18, 23, 28, 33

b) Use technology or grid paper to graph the sequence in part a. Plot the *Term value* on the vertical axis and the *Term number* on the horizontal axis. Print the graph or sketch it on this grid.

i) How do you know that you have graphed a linear function?

The points lie on a non-vertical straight line.

ii) What does the slope of the line through the points represent? Explain why.

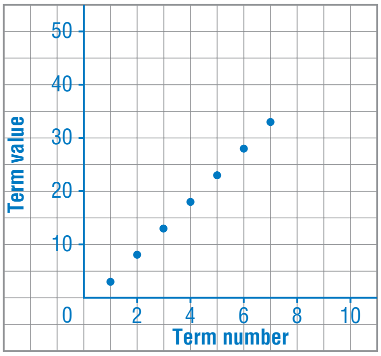
The slope is the common difference because it is the rise when the run is 1; that is, after the first point, each point can be plotted by moving 5 units up and 1 unit right.

TEACHER NOTE

Achievement Indicator

Question 9 addresses AI 9.2: Provide and justify an example of an arithmetic sequence.

Sample response



TEACHER NOTE

In question 9, some students may think that 0 cannot be a common difference; however, the definition of an arithmetic sequence does not preclude $d = 0$.

10. Two terms of an arithmetic sequence are given. Determine the indicated terms.

a) $t_4 = 24$, $t_{10} = 66$; determine t_1

$t_{10} = t_4 + 6d$
Substitute for t_{10} and t_4, then solve for d.
$66 = 24 + 6d$
$6d = 42$
$d = 7$
$t_1 = t_4 - 3d$
Substitute for t_4 and d.
$t_1 = 24 - 3(7)$
$t_1 = 3$

b) $t_3 = 81$, $t_{12} = 27$; determine t_{23}

$t_{12} = t_3 + 9d$
Substitute for t_{12} and t_3.
$27 = 81 + 9d$
$9d = -54$
$d = -6$
$t_{23} = t_{12} + 11d$
Substitute for t_{12} and d.
$t_{23} = 27 + 11(-6)$
$t_{23} = -39$

TEACHER NOTE

Achievement Indicator

Questions 10 and 13 address AI 9.5:
Determine t_1, d, n, or t_n in a problem that involves an arithmetic sequence.

11. Create an arithmetic sequence for each description below. For each sequence, write the first 6 terms and a rule for t_n.

a) an increasing sequence

Sample response:
Choose a positive common difference.
Use: $t_1 = 4$ and $d = 3$
The sequence is:
4, 7, 10, 13, 16, 19, . . .
Use: $t_n = t_1 + d(n - 1)$
Substitute: $t_1 = 4$, $d = 3$
$t_n = 4 + 3(n - 1)$
$t_n = 1 + 3n$

b) a decreasing sequence

Choose a negative common difference.
Use: $t_1 = 4$ and $d = -3$
The sequence is:
4, 1, −2, −5, −8, −11, . . .
Use: $t_n = t_1 + d(n - 1)$
Substitute: $t_1 = 4$, $d = -3$
$t_n = 4 - 3(n - 1)$
$t_n = 7 - 3n$

c) every term is negative

Sample response:

Choose a negative first term, and a negative common difference.
Use: $t_1 = -2$ and $d = -3$
The sequence is:
$-2, -5, -8, -11, -14, -17, \ldots$
Use: $t_n = t_1 + d(n - 1)$
Substitute: $t_1 = -2$, $d = -3$
$t_n = -2 - 3(n - 1)$
$t_n = 1 - 3n$

d) every term is an even number

Choose an even first term and an even common difference.
Use: $t_1 = -2$ and $d = -4$
The sequence is:
$-2, -6, -10, -14, -18, -22, \ldots$
Use: $t_n = t_1 + d(n - 1)$
Substitute: $t_1 = -2$, $d = -4$
$t_n = -2 - 4(n - 1)$
$t_n = 2 - 4n$

12. Claire wrote the first 3 terms of an arithmetic sequence: 3, 6, 9, . . .
When she asked Alex to extend the sequence to the first 10 terms, he wrote:
3, 6, 9, 3, 6, 9, 3, 6, 9, 3, . . .

a) Is Alex correct? Explain.

No, Alex's sequence is not arithmetic because the terms do not increase or decrease by the same number.

b) What fact did Alex ignore when he extended the sequence?

Alex did not use the common difference of 3 to calculate each term.

c) What is the correct sequence?

Add 3 to get each next term: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, . . .

TEACHER NOTE

Achievement Indicator
Question 12 addresses AI 9.1: Identify the assumption(s) made when defining an arithmetic sequence.

13. Determine whether 100 is a term of an arithmetic sequence with $t_3 = 250$ and $t_6 = 245.5$.

Let the common difference be d.
Use: $t_6 = t_3 + 3d$ Substitute: $t_6 = 245.5$, $t_3 = 250$
$245.5 = 250 + 3d$
$3d = -4.5$
$d = -1.5$
Use: $t_1 = t_3 - 2d$ Substitute: $t_3 = 250$, $d = -1.5$
$t_1 = 250 - 2(-1.5)$
$t_1 = 253$
Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_n = 100$, $d = -1.5$, $t_1 = 253$
$100 = 253 - 1.5(n - 1)$
$1.5n = 154.5$
$n = 103$
Since $t_{103} = 100$, then 100 is a term of the sequence.

14. The Chinese zodiac associates years with animals. Ling was born in 1994, the Year of the Dog.

a) The Year of the Dog repeats every 12 years. List the first three years that Ling will celebrate her birthday in the Year of the Dog.

Add 12 to 1994 three times: 2006, 2018, 2030

b) Why do the years in part a form an arithmetic sequence?

The difference between consecutive dates is constant.

c) In 2099, Nunavut will celebrate its 100th birthday.

Will that year also be the Year of the Dog? Explain.

All terms in the sequence are even numbers;
2099 is an odd number so 2099 cannot be the Year of the Dog.

15. In this arithmetic sequence: 3, 8, 13, 18, . . . ; which term has the value 123?

d is: $8 - 3 = 5$
Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_n = 123$, $t_1 = 3$, $d = 5$
$123 = 3 + 5(n - 1)$
$123 = 3 + 5n - 5$
$5n = 125$
$n = 25$
123 is the 25th term.

16. For two different arithmetic sequences, $t_5 = -1$. What are two possible sequences? Explain your reasoning.

Sample response: For one sequence, choose a number for the common difference, d, such as 5. Keep subtracting 5 to get the preceding terms.

$t_5 = -1$ $\quad t_4 = -1 - 5 = -6$ $\quad t_3 = -6 - 5 = -11$ $\quad t_2 = -11 - 5 = -16$ $\quad t_1 = -16 - 5 = -21$

One arithmetic sequence is: −21, −16, −11, −6, −1, . . .

For the other sequence, choose a different number for d, such as −8. Keep subtracting −8.

$t_5 = -1$ $\quad t_4 = -1 + 8 = 7$ $\quad t_3 = 7 + 8 = 15$ $\quad t_2 = 15 + 8 = 23$ $\quad t_1 = 23 + 8 = 31$

Another arithmetic sequence is: 31, 23, 15, 7, −1, . . .

TEACHER NOTE

Achievement Indicator

Question 14 addresses AI 9.8: Solve a problem that involves an arithmetic sequence.

C

17. A sequence is created by adding each term of an arithmetic sequence to the preceding term.

a) Show that the new sequence is arithmetic.

Use the general sequence: $t_1, t_1 + d, t_1 + 2d, t_1 + 3d, t_1 + 4d, \ldots$
The new sequence is: $t_1 + t_1 + d, t_1 + d + t_1 + 2d, t_1 + 2d + t_1 + 3d,$
$t_1 + 3d + t_1 + 4d, \ldots$
This simplifies to: $2t_1 + d, 2t_1 + 3d, 2t_1 + 5d, 2t_1 + 7d, \ldots$
This sequence has first term $2t_1 + d$ and common difference $2d$, so the sequence is arithmetic.

b) How are the common differences of the two sequences related?

The common difference of the new sequence is double the common difference of the original sequence.

18. In this arithmetic sequence, k is a natural number: $k, \frac{2k}{3}, \frac{k}{3}, 0, \ldots$

a) Determine t_6.

The common difference, d, is: $\frac{2k}{3} - k = -\frac{k}{3}$

$t_4 = 0 \qquad t_5 = 0 + \left(-\frac{k}{3}\right) \qquad t_6 = -\frac{k}{3} - \frac{k}{3}$

$\qquad\qquad\quad = -\frac{k}{3} \qquad\qquad\quad = -\frac{2k}{3}$

t_6 is $-\frac{2k}{3}$.

b) Write an expression for t_n.

Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_1 = k, d = -\frac{k}{3}$

$t_n = k + \left(-\frac{k}{3}\right)(n - 1)$

$t_n = k - \frac{kn}{3} + \frac{k}{3}$

$t_n = \frac{4k}{3} - \frac{kn}{3}$

c) Suppose $t_{20} = -16$; determine the value of k.

Use: $t_n = \frac{4k}{3} - \frac{kn}{3}$ Substitute: $t_n = -16, n = 20$

$-16 = \frac{4k}{3} - \frac{20k}{3}$

$-16 = -\frac{16k}{3}$

$k = 3$

Multiple-Choice Questions

1. How many of these sequences have a common difference of -4?
$-19, -15, -11, -7, -3, \ldots$
$19, 15, 11, 7, 3, \ldots$
$3, 7, 11, 15, 19, \ldots$
$-3, 7, -11, 15, -19, \ldots$

A. 0 **B.** 1 **C.** 2 **D.** 3

2. Which number below is a term of this arithmetic sequence?
$97, 91, 85, 79, 73, \ldots$

A. -74 **B.** -75 **C.** -76 **D.** -77

3. The first 6 terms of an arithmetic sequence are plotted on a grid. The coordinates of two points on the graph are (3, 11) and (6, 23). What is an expression for the general term of the sequence?

A. $6n - 3$ **B.** $3n + 11$ **C.** $4n - 1$ **D.** $1 + 4n$

TEACHER NOTE

Solution strategy: Determine the slope of the graph (4), then look for a pattern in the coordinates that includes multiples of 4.

Study Note

How are arithmetic sequences and linear functions related?

Both an arithmetic sequence and a linear function show constant change. An arithmetic sequence can be described by a linear function whose domain is the natural numbers.

TEACHER NOTE

Achievement Indicator

The Study Note addresses AI 9.4: Describe the relationship between arithmetic sequences and linear functions.

ANSWERS

4. a) 4 **b)** -2 **c)** 7 **5. a)** 3; 21, 24, 27 **b)** -4; 13, 9, 5 **6. a)** 36 **b)** -15
7. a) $-3, 1, 5, 9$ **b)** $-0.5, -2, -3.5, -5$ **10. a)** 3 **b)** -39
12. a) no **c)** 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, . . . **13.** 100 is a term.
14. a) 2006, 2018, 2030 **c)** no **15.** t_{25} **18. a)** $-\frac{2k}{3}$ **b)** $\frac{4k}{3} - \frac{kn}{3}$ **c)** 3

Multiple Choice

1. B **2.** D **3.** C

1.2 Arithmetic Series

Lesson Organizer
75 min

Key Math Concepts
An arithmetic series is the sum of the terms in an arithmetic sequence.

Curriculum Focus

SO	AI
RF9	9.1, 9.6, 9.7, 9.8

Processes: CN, PS, R, T

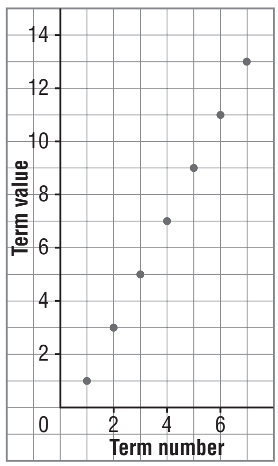
Student Materials
- scientific calculator
- graphing calculator (*optional*)

Vocabulary
series, arithmetic series, partial sums

FOCUS **Derive a rule to determine the sum of *n* terms of an arithmetic series, then solve related problems.**

Get Started

Suppose this sequence continues.
What is the value of the 8th term?



The 7th term is 13, the common difference is 2, so the 8th term is 15.

What is an expression for the nth term?

Each term value is 1 less than 2 times the term number, so the nth term is $2n - 1$.

Is 50 a term in this sequence? How do you know?

No, because 50 is an even number and all the term values are odd numbers.

Construct Understanding

Talise displayed 90 photos of the Regina Dragon Boat Festival in 5 rows. The difference between the numbers of photos in consecutive rows was constant.
How many different sequences are possible? Justify your answer.

Since the differences in the numbers of photos in consecutive rows are constant, these numbers form an arithmetic sequence.

Suppose the common difference is 0. Then the rows would have the same numbers of photos, so the number of photos in each row would be: $\frac{90}{5} = 18$

Suppose the common difference is 1. Start with 18 photos in the 3rd (middle) row. Then, the 4th row has $18 + 1 = 19$; and the 5th row has $19 + 1 = 20$. The 2nd row has $18 - 1 = 17$; and the 1st row has $17 - 1 = 16$

For each possible sequence, there are 18 photos in the 3rd row.

So, add and subtract possible common differences to get the numbers of photos in the other rows.

Difference	Number of Photos in Rows 1 to 5	Difference	Number of Photos in Rows 1 to 5
−8	34, 26, 18, 10, 2	1	16, 17, 18, 19, 20
−7	32, 25, 18, 11, 4	2	14, 16, 18, 20, 22
−6	30, 24, 18, 12, 6	3	12, 15, 18, 21, 24
−5	28, 23, 18, 13, 8	4	10, 14, 18, 22, 26
−4	26, 22, 18, 14, 10	5	8, 13, 18, 23, 28
−3	24, 21, 18, 15, 12	6	6, 12, 18, 24, 30
−2	22, 20, 18, 16, 14	7	4, 11, 18, 25, 32
−1	20, 19, 18, 17, 16	8	2, 10, 18, 26, 34
0	18, 18, 18, 18, 18		

So, 17 different sequences are possible.

A **series** is a sum of the terms in a sequence.
An **arithmetic series** is the sum of the terms in an arithmetic sequence.
For example, an arithmetic sequence is: 5, 8, 11, 14, . . .
The related arithmetic series is: $5 + 8 + 11 + 14 + \ldots$
The term, S_n, is used to represent the sum of the first n terms of a series.
The nth term of an arithmetic series is the nth term of the related arithmetic sequence.
For the arithmetic series above:

$$\begin{array}{llll} S_1 = t_1 & S_2 = t_1 + t_2 & S_3 = t_1 + t_2 + t_3 & S_4 = t_1 + t_2 + t_3 + t_4 \\ S_1 = 5 & S_2 = 5 + 8 & S_3 = 5 + 8 + 11 & S_4 = 5 + 8 + 11 + 14 \\ & S_2 = 13 & S_3 = 24 & S_4 = 38 \end{array}$$

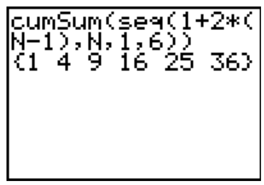
These are called *partial sums*.

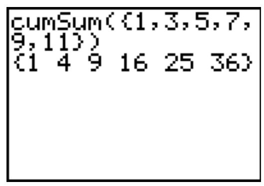
If there are only a few terms, S_n can be determined using mental math.
To develop a rule to determine S_n, use algebra.
Write the sum on one line, reverse the order of the terms on the next line, then add vertically. Write the sum as a product.

TECHNOLOGY NOTE

Graphing Calculator

To determine partial sums of an arithmetic series on a graphing calculator, use the cumulative sum function and generate the terms using the sequence function or simply list the terms. For example, to determine S_n for a series with $t_n = 1 + 2(n - 1)$:





Animation

Extra Material

A pictorial development that supports the derivation of the sum formula

THINK FURTHER

Why can the $(n - 1)$th term be written as $t_n - d$?

The $(n - 1)$th term comes immediately before the nth term. So, the value of the $(n - 1)$th term is the value of the nth term minus the common difference.

$$\begin{aligned} S_n &= t_1 + (t_1 + d) + (t_1 + 2d) + \ldots + (t_n - d) + t_n \\ + \quad S_n &= t_n + (t_n - d) + (t_n - 2d) + \ldots + (t_1 + d) + t_1 \\ \hline 2S_n &= (t_1 + t_n) + (t_1 + t_n) + (t_1 + t_n) + \ldots + (t_1 + t_n) + (t_1 + t_n) \\ 2S_n &= n(t_1 + t_n) \quad \text{Solve for } S_n. \\ S_n &= \frac{n(t_1 + t_n)}{2} \end{aligned}$$

The rule above is used when t_1 and t_n are known. Substitute for t_n to write S_n in a different way, so it can be used when t_1 and the common difference, d, are known.

$$S_n = \frac{n(t_1 + t_n)}{2} \qquad \text{Substitute: } t_n = t_1 + d(n - 1)$$

$$S_n = \frac{n[t_1 + t_1 + d(n - 1)]}{2} \qquad \text{Combine like terms.}$$

$$S_n = \frac{n[2t_1 + d(n - 1)]}{2}$$

The Sum of *n* Terms of an Arithmetic Series

For an arithmetic series with 1st term, t_1, common difference, d, and nth term, t_n, the sum of the first n terms, S_n, is:

$$S_n = \frac{n(t_1 + t_n)}{2} \quad \text{or} \quad S_n = \frac{n[2t_1 + d(n - 1)]}{2}$$

Example 1 **Determining the Sum, Given the Series**

Determine the sum of the first 6 terms of this arithmetic series:
$-75 - 69 - 63 - 57 - 51 - 45 - \ldots$

SOLUTION

$-75 - 69 - 63 - 57 - 51 - 45 - \ldots$
t_1 is -75 and t_6 is -45.

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute: $n = 6$, $t_1 = -75$, $t_n = -45$

$$S_6 = \frac{6(-75 - 45)}{2}$$
$$S_6 = -360$$

The sum of the first 6 terms is -360.

Check Your Understanding

1. Determine the sum of the first 6 terms of this arithmetic series: $25 + 14 + 3 - 8 - 19 - 30 - \ldots$

Use: $S_n = \frac{n(t_1 + t_n)}{2}$

Substitute: $n = 6$, $t_1 = 25$, $t_n = -30$

$S_6 = \frac{6(25 - 30)}{2}$

$S_6 = -15$

Check Your Understanding

Answer:

1. -15

Example 2	Determining the Sum, Given the First Term and Common Difference

An arithmetic series has $t_1 = 5.5$ and $d = -2.5$; determine S_{40}.

SOLUTION

Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$ Substitute: $n = 40$, $t_1 = 5.5$, $d = -2.5$

$$S_{40} = \frac{40[2(5.5) - 2.5(40 - 1)]}{2}$$
$$S_{40} = -1730$$

Check Your Understanding

2. An arithmetic series has $t_1 = 3$ and $d = -4$; determine S_{25}.

Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$

Substitute: $n = 25$, $t_1 = 3$, $d = -4$

$$S_{25} = \frac{25[2(3) - 4(25 - 1)]}{2}$$
$$S_{25} = -1125$$

Example 3	Determining the First Few Terms Given the Sum, Common Difference, and One Term

An arithmetic series has $S_{20} = 143\frac{1}{3}$, $d = \frac{1}{3}$, and $t_{20} = 10\frac{1}{3}$; determine the first 3 terms of the series.

SOLUTION

S_{20} and t_{20} are known, so use this rule to determine t_1:

$S_n = \dfrac{n(t_1 + t_n)}{2}$ Substitute: $n = 20$, $S_{20} = 143\frac{1}{3}$, $t_{20} = 10\frac{1}{3}$

$143\frac{1}{3} = \dfrac{20\left(t_1 + 10\frac{1}{3}\right)}{2}$ Simplify.

$143\frac{1}{3} = 10\left(t_1 + 10\frac{1}{3}\right)$

$143\frac{1}{3} = 10t_1 + 103\frac{1}{3}$ Solve for t_1.

$40 = 10t_1$

$4 = t_1$

The first term is 4 and the common difference is $\frac{1}{3}$.

So, the first 3 terms of the series are written as the partial sum:

$4 + 4\frac{1}{3} + 4\frac{2}{3}$

Check Your Understanding

3. An arithmetic series has $S_{15} = 93.75$, $d = 0.75$, and $t_{15} = 11.5$; determine the first 3 terms of the series.

S_{15} and t_{15} are known, so use this rule:

$$S_n = \frac{n(t_1 + t_n)}{2}$$

Substitute: $n = 15$, $S_{15} = 93.75$, $t_{15} = 11.5$

$$93.75 = \frac{15(t_1 + 11.5)}{2}$$
$$93.75 = 7.5(t_1 + 11.5)$$
$$93.75 = 7.5t_1 + 86.25$$
$$7.5 = 7.5t_1$$
$$1 = t_1$$

The 1st term is 1.

The 2nd term is: $1 + 0.75 = 1.75$

The 3rd term is: $1.75 + 0.75 = 2.5$

So, the first 3 terms of the series are: $1 + 1.75 + 2.5$

THINK FURTHER

In *Example 3*, which partial sums are natural numbers? Why?

S_1 is a natural number, and $t_2 + t_3$ is a natural number, so $S_3 = S_1 + t_2 + t_3$ is a natural number. Since t_4 is a natural number, then $S_4 = S_3 + t_4$ is a natural number. Since $t_5 + t_6$ is a natural number, then $S_6 = S_4 + t_5 + t_6$ is a natural number. This pattern continues. The partial sums that are natural numbers are: $S_1, S_3, S_4, S_6, S_7, S_9, S_{10}$, and so on

Check Your Understanding

Answers:

2. –1125

3. 1 + 1.75 + 2.5

Animation

Example 4 Using an Arithmetic Series to Model and Solve a Problem

Students created a trapezoid from the cans they had collected for the food bank. There were 10 rows in the trapezoid. The bottom row had 100 cans. Each consecutive row had 5 fewer cans than the previous row. How many cans were in the trapezoid?

SOLUTION

The numbers of cans in the rows form an arithmetic sequence with first 3 terms 100, 95, 90, . . .
The total number of cans is the sum of the first 10 terms of the arithmetic series:
$100 + 95 + 90 + \dots$

Use: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$ Substitute: $n = 10$, $t_1 = 100$, $d = -5$

$$S_{10} = \frac{10[2(100) - 5(10 - 1)]}{2}$$

$$S_{10} = 775$$

There were 775 cans in the trapezoid.

Check Your Understanding

4. The bottom row in a trapezoid had 49 cans. Each consecutive row had 4 fewer cans than the previous row. There were 11 rows in the trapezoid. How many cans were in the trapezoid?

The numbers of cans in the rows form an arithmetic sequence with first 3 terms 49, 45, 41, . . .

Use: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$

Substitute: $n = 11$, $t_1 = 49$, $d = -4$

$S_{11} = \frac{11[2(49) - 4(11 - 1)]}{2}$

$S_{11} = 319$

There were 319 cans in the trapezoid.

Discuss the Ideas

1. How are an arithmetic series and an arithmetic sequence related?

The terms of an arithmetic series form an arithmetic sequence. The terms of an arithmetic sequence are added to form an arithmetic series.

2. Suppose you know the 1st and nth terms of an arithmetic series. What other information do you need to determine the value of n?

When I know t_1 and t_n, I need to know the common difference, d, then I can use the rule $t_n = t_1 + d(n - 1)$ to determine n. Or, when I know t_1 and t_n, I need to know the sum of the first n terms, S_n, then I can use the rule $S_n = \frac{n(t_1 + t_n)}{2}$ to determine n.

TEACHER NOTE

DI: Common Difficulties

Some students may have difficulty recalling the sum formula that involves the first term and common difference of an arithmetic series, as applied in *Example 2*. They could use the given common difference to determine the nth term of the series, then use the formula:

$S_n = \frac{n(t_1 + t_n)}{2}$

Check Your Understanding

Answer:

4. 319 cans

Exercises

A

3. Use each arithmetic sequence to write the first 4 terms of an arithmetic series.

a) 2, 4, 6, 8, . . .

$2 + 4 + 6 + 8$

b) −2, 3, 8, 13, . . .

$-2 + 3 + 8 + 13$

c) 4, 0, −4, −8, . . .

$4 + 0 - 4 - 8$

d) $\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}, \ldots$

$\frac{1}{2} + \frac{1}{4} + 0 - \frac{1}{4}$

4. Determine the sum of the given terms of each arithmetic series.

a) 12 + 10 + 8 + 6 + 4

$12 + 10 + 8 + 6 + 4 = 40$

b) −2 − 4 − 6 − 8 − 10

$-2 - 4 - 6 - 8 - 10 = -30$

5. Determine the sum of the first 20 terms of each arithmetic series.

a) 3 + 7 + 11 + 15 + . . .

Use: $S_n = \frac{n[2t_1 + d(n-1)]}{2}$

Substitute:

$n = 20, t_1 = 3, d = 4$

$S_{20} = \frac{20[2(3) + 4(20-1)]}{2}$

$S_{20} = 820$

b) −21 − 15.5 − 10 − 4.5 − . . .

Use: $S_n = \frac{n[2t_1 + d(n-1)]}{2}$

Substitute:

$n = 20, t_1 = -21, d = 5.5$

$S_{20} = \frac{20[2(-21) + 5.5(20-1)]}{2}$

$S_{20} = 625$

B

6. For each arithmetic series, determine the indicated value.

a) −4 − 11 − 18 − 25 − . . . ; determine S_{28}

Use: $S_n = \frac{n[2t_1 + d(n-1)]}{2}$

Substitute:

$n = 28, t_1 = -4, d = -7$

$S_{28} = \frac{28[2(-4) - 7(28-1)]}{2}$

$S_{28} = -2758$

b) 1 + 3.5 + 6 + 8.5 + . . . ; determine S_{42}

Use: $S_n = \frac{n[2t_1 + d(n-1)]}{2}$

Substitute:

$n = 42, t_1 = 1, d = 2.5$

$S_{42} = \frac{42[2(1) + 2.5(42-1)]}{2}$

$S_{42} = 2194.5$

7. Use the given data about each arithmetic series to determine the indicated value.

a) $S_{20} = -850$ and $t_{20} = -90$; determine t_1

Use: $S_n = \dfrac{n(t_1 + t_n)}{2}$

Substitute:

$S_n = -850$, $n = 20$, $t_n = -90$

$$-850 = \frac{20(t_1 - 90)}{2}$$
$$-850 = 10t_1 - 900$$
$$50 = 10t_1$$
$$t_1 = 5$$

b) $S_{15} = 322.5$ and $t_1 = 4$; determine d

Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$

Substitute:

$S_n = 322.5$, $n = 15$, $t_1 = 4$

$$322.5 = \frac{15[2(4) + d(15 - 1)]}{2}$$
$$322.5 = 7.5(8 + 14d)$$
$$43 = 8 + 14d$$
$$35 = 14d$$
$$d = 2.5$$

c) $S_n = -126$, $t_1 = -1$, and $t_n = -20$; determine n

Use: $S_n = \dfrac{n(t_1 + t_n)}{2}$

Substitute:

$S_n = -126$, $t_1 = -1$, $t_n = -20$

$$-126 = \frac{n(-1 - 20)}{2}$$
$$-252 = -21n$$
$$n = 12$$

d) $t_1 = 1.5$ and $t_{20} = 58.5$; determine S_{15}

Use $t_n = t_1 + d(n - 1)$ to determine d.

Substitute: $t_n = 58.5$, $t_1 = 1.5$, $n = 20$

$$58.5 = 1.5 + d(20 - 1)$$
$$57 = 19d$$
$$d = 3$$

Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$

Substitute: $n = 15$, $t_1 = 1.5$, $d = 3$

$$S_{15} = \frac{15[2(1.5) + 3(15 - 1)]}{2}$$
$$S_{15} = \frac{15(3 + 42)}{2}$$
$$S_{15} = 337.5$$

8. Two hundred seventy-six students went to a powwow. The first bus had 24 students. The numbers of students on the buses formed an arithmetic sequence. What additional information do you need to determine the number of buses? Explain your reasoning.

I need to know the number of students on the last bus, then I can use the rule $S_n = \dfrac{n(t_1 + t_n)}{2}$. Or, I need to know the common difference, d, then I can use the rule $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$. In each rule, I substitute what I know, then solve for n.

TEACHER NOTE

Achievement Indicator

Question 7 addresses AI 9.7: Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.

9. Ryan's grandparents loaned him the money to purchase a BMX bike. He agreed to repay \$25 at the end of the first month, \$30 at the end of the second month, \$35 at the end of the third month, and so on. Ryan repaid the loan in 12 months. How much did the bike cost? How do you know your answer is correct?

Ryan's repayments form an arithmetic series with 12 terms, where the 1st term is his first payment, and the common difference is \$5.

Use: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$ **Substitute:** $n = 12$, $t_1 = 25$, $d = 5$

$S_{12} = \frac{12[2(25) + 5(12 - 1)]}{2}$

$S_{12} = 6(50 + 55)$

$S_{12} = 630$

The bike cost \$630.

I used a calculator to add the 12 payments to check that the answer is the same.

TEACHER NOTE

Achievement Indicator

Question 9 addresses AI 9.8: Solve a problem that involves an arithmetic series.

10. Determine the sum of the indicated terms of each arithmetic series.

a) $31 + 35 + 39 + \ldots + 107$

Use $t_n = t_1 + d(n - 1)$ **to determine** n**. Substitute:**

$t_n = 107$, $t_1 = 31$, and $d = 4$

$107 = 31 + 4(n - 1)$

$76 = 4n - 4$

$80 = 4n$

$n = 20$

Use: $S_n = \frac{n(t_1 + t_n)}{2}$

Substitute:

$n = 20$, $t_1 = 31$, and $t_n = 107$

$S_{20} = \frac{20(31 + 107)}{2}$

$S_{20} = 1380$

b) $-13 - 10 - 7 - \ldots + 62$

Use $t_n = t_1 + d(n - 1)$ **to determine** n**. Substitute:**

$t_n = 62$, $t_1 = -13$, and $d = 3$

$62 = -13 + 3(n - 1)$

$75 = 3n - 3$

$78 = 3n$

$n = 26$

Use: $S_n = \frac{n(t_1 + t_n)}{2}$

Substitute:

$n = 26$, $t_1 = -13$, and $t_n = 62$

$S_{26} = \frac{26(-13 + 62)}{2}$

$S_{26} = 637$

11. a) Explain how this series could be arithmetic.

$1 + 3 + \ldots$

This series could be arithmetic if each term was calculated by adding 2 to the preceding term.

b) What information do you need to be certain that this is an arithmetic series?

I need to know that the number added each time is 2.

TEACHER NOTE

Achievement Indicator

Question 11 addresses AI 9.1: Identify the assumption(s) made when defining an arithmetic series.

12. An arithmetic series has $S_{10} = 100$, $t_1 = 1$, and $d = 2$. How can you use this information to determine S_{11} without using a rule for the sum of an arithmetic series? What is S_{11}?

$S_{11} = S_{10} + t_{11}$
I use $t_n = t_1 + d(n - 1)$ to determine t_{11}.
I substitute $n = 11$, $t_1 = 1$, $d = 2$.
$t_{11} = 1 + 2(11 - 1)$
$t_{11} = 21$
Then $S_{11} = 100 + 21$
$S_{11} = 121$

13. The side lengths of a quadrilateral form an arithmetic sequence. The perimeter is 74 cm. The longest side is 29 cm. What are the other side lengths?

The sum of the side lengths form an arithmetic series with 4 terms, where $t_4 = 29$ and $S_4 = 74$.

Use $S_n = \frac{n(t_1 + t_n)}{2}$ to determine t_1.
Substitute: $S_n = 74$, $n = 4$, $t_n = 29$

$74 = \frac{4(t_1 + 29)}{2}$
$74 = 2t_1 + 58$
$16 = 2t_1$
$t_1 = 8$

Use $t_n = t_1 + d(n - 1)$ to determine d.
Substitute: $t_n = 29$, $t_1 = 8$, $n = 4$
$29 = 8 + d(4 - 1)$
$21 = 3d$
$d = 7$
So, the other side lengths are: 8 cm, 15 cm, and 22 cm

14. Derive a rule for the sum of the first n natural numbers:
$1 + 2 + 3 + \ldots + n$

The sum of the numbers is an arithmetic series with $t_1 = 1$, $d = 1$, and $t_n = n$.

Use: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$ Substitute: $t_1 = 1$, $d = 1$

$S_n = \frac{n[2(1) + 1(n - 1)]}{2}$

$S_n = \frac{n(n + 1)}{2}$

TEACHER NOTE

Achievement Indicator

Question 14 addresses AI 9.6: Derive a rule for determining the sum of n terms of an arithmetic series.

15. The sum of the first 5 terms of an arithmetic series is 170. The sum of the first 6 terms is 225. The common difference is 7. Determine the first 4 terms of the series.

$S_5 = 170, S_6 = 225$
$t_6 = S_6 - S_5$
$= 225 - 170$
$= 55$
Use $t_n = t_1 + d(n - 1)$ to determine t_1.
Substitute: $t_n = 55, d = 7, n = 6$
$55 = t_1 + 7(6 - 1)$
$t_1 = 20$
So, $t_2 = 27$, $t_3 = 34$, and $t_4 = 41$
The first 4 terms are: $20 + 27 + 34 + 41$

C

16. The sum of the first n terms of an arithmetic series is: $S_n = 3n^2 - 8n$
Determine the first 4 terms of the series.

Determine S_1, S_2, S_3, and S_4.
In $S_n = 3n^2 - 8n$:

Substitute: $n = 1$
$S_1 = 3(1)^2 - 8(1)$
$= -5$

Substitute: $n = 2$
$S_2 = 3(2)^2 - 8(2)$
$= -4$

Substitute: $n = 3$
$S_3 = 3(3)^2 - 8(3)$
$= 3$

Substitute: $n = 4$
$S_4 = 3(4)^2 - 8(4)$
$= 16$

$t_1 = S_1$
$= -5$

$t_2 = S_2 - S_1$
$= 1$

$t_3 = S_3 - S_2$
$= 7$

$t_4 = S_4 - S_3$
$= 13$

17. Each number from 1 to 60 is written on one of 60 index cards. The cards are arranged in rows with equal lengths, and no cards are left over. The sum of the numbers in each row is 305. How many rows are there?

Determine the sum of the first 60 natural numbers:
$1 + 2 + 3 + \ldots + 60$
This is an arithmetic series with $t_1 = 1$, $d = 1$, and $t_{60} = 60$.
Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute: $n = 60$, $t_1 = 1$, $t_n = 60$
$S_n = \frac{60(1 + 60)}{2}$
$S_n = 1830$
The sum of the numbers in each row is 305, so the number of rows is:
$\frac{1830}{305} = 6$

18. Determine the arithmetic series that has these partial sums: $S_4 = 26$, $S_5 = 40$, and $S_6 = 57$

$t_6 = S_6 - S_5$	$t_5 = S_5 - S_4$	$d = t_6 - t_5$
$t_6 = 57 - 40$	$t_5 = 40 - 26$	$d = 17 - 14$
$= 17$	$= 14$	$= 3$
t_4 is: $t_5 - 3 = 11$	t_3 is: $t_4 - 3 = 8$	t_2 is: $t_3 - 3 = 5$

t_1 is: $t_2 - 3 = 2$

The arithmetic series is: $2 + 5 + 8 + 11 + 14 + 17 + \dots$

Multiple-Choice Questions

1. Which of these series could be arithmetic?

A. $2.5 + 5 + 7.5 + 11 + \dots$ **B.** $-2.5 - 5 - 7.5 - 11 - \dots$

C. $3.5 + 6 + 8.5 + 11 + \dots$ **D.** $3.5 - 6 - 8.5 - 11 + \dots$

2. For which arithmetic series below is 115 the sum of the first 10 terms?

A. $39 + 34 + 29 + 24 + 19 + \dots$

B. $-11 - 6 - 1 + 4 + 9 + \dots$

C. $11 + 6 + 1 - 4 - 9 - \dots$

D. $39 + 31 + 23 + 15 + 7 + \dots$

3. How many of these expressions could be used to determine the sum to n terms of an arithmetic series?

$\frac{n[2t_1 + d(n + 1)]}{2}$ $\frac{n[2t_1 - d(n + 1)]}{2}$ $\frac{n(t_1 + t_n)}{2}$ $\frac{n(t_1 - t_n)}{2}$

A. 4 **B.** 3 **C.** 2 **D.** 1

TEACHER NOTE

Solution strategy: For each series, calculate the difference of each pair of consecutive terms to see which differences are constant.

Study Note

There are two forms of the rule to determine the sum of the first n terms of an arithmetic series. When would you use each form of the rule?

I would use $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$ when I know the number of terms, the first term, and the common difference. I would use $S_n = \frac{n(t_1 + t_n)}{2}$ when I know the number of terms, the first term, and the nth term.

ANSWERS

3. a) $2 + 4 + 6 + 8$ **b)** $-2 + 3 + 8 + 13$ **c)** $4 + 0 - 4 - 8$ **d)** $\frac{1}{2} + \frac{1}{4} + 0 - \frac{1}{4}$ **4. a)** 40 **b)** -30 **5. a)** 820 **b)** 625 **6. a)** -2758 **b)** 2194.5 **7. a)** 5 **b)** 2.5 **c)** 12 **d)** 337.5 **9.** \$630 **10. a)** 1380 **b)** 637 **12.** 121 **13.** 8 cm, 15 cm, and 22 cm **14.** $S_n = \frac{n(n + 1)}{2}$ **15.** $20 + 27 + 34 + 41$ **16.** $-5 + 1 + 7 + 13$ **17.** 6 **18.** $2 + 5 + 8 + 11 + 14 + 17 + \dots$

Multiple Choice

1. C **2.** B **3.** D

CHECKPOINT 1

Self-Assess

Can you . . .	To check, try question . . .	For review, see . . .
write an example of an arithmetic sequence and explain how you know it is arithmetic?	2	Page 4 in Lesson 1.1 (Example 1)
explain the meaning of the symbols n, t_1, t_n, and d?		Page 3 in Lesson 1.1
identify the assumptions made when defining an arithmetic sequence or series?		Page 7 in Lesson 1.1
use a rule to determine the nth term in an arithmetic sequence?	3	Page 4 in Lesson 1.1
describe the relationship between an arithmetic sequence and a linear function?		Page 3 in Lesson 1.1
use a rule to determine t_n and n in an arithmetic sequence given the values of t_1 and d?	3b, 4b	Page 5 in Lesson 1.1 (Example 2)
use a rule to determine t_1 and d in an arithmetic sequence given the values of t_n and n?	4a	Page 6 in Lesson 1.1 (Example 3)
solve problems involving arithmetic sequences?	5	Pages 6-7 in Lesson 1.1 (Example 4)
use a rule to determine the sum S_n of an arithmetic series?		Page 16 in Lesson 1.2
use a rule to determine S_n in an arithmetic series given the values of n, t_1, and t_n?	7b, 8	Page 16 in Lesson 1.2 (Example 1)
use a rule to determine S_n in an arithmetic series given the values of n, t_1, and d?		Page 17 in Lesson 1.2 (Example 2)
use a rule to determine t_1 in an arithmetic series given the values of S_n, t_n, and d?	9a	Page 17 in Lesson 1.2 (Example 3)
solve problems involving arithmetic series?		Page 18 in Lesson 1.2 (Example 4)

TEACHER NOTE

Have students complete the mind map on Master 1.3a to summarize their knowledge of arithmetic series.

Assess Your Understanding

1.1

1. **Multiple Choice** Which arithmetic sequence has $d = -8$ and $t_{10} = -45$?

A. 27, 19, 11, 3, . . . **B.** −8, −12, −16, −20, . . .

C. −5, −13, −21, −29, . . . **D.** −27, −19, −11, −3, . . .

2. Write the first 4 terms of an arithmetic sequence with its 5th term equal to −4.

Sample response: I chose a common difference of 2.
$t_5 = -4$; so t_4 is $-4 - 2 = -6$; t_3 is $-6 - 2 = -8$; t_2 is $-8 - 2 = -10$; and t_1 is $-10 - 2 = -12$
My arithmetic sequence is: −12, −10, −8, −6, . . .

TEACHER NOTE

Achievement Indicators

Question 2 addresses AI 9.2: Provide and justify an example of an arithmetic sequence.

Question 3 addresses AI 9.3: Derive a rule for determining the general term of an arithmetic sequence.

Question 4 addresses AI 9.5: Determine t_1, d, n, or t_n in a problem that involves an arithmetic sequence.

3. This sequence is arithmetic: −8, −11, −14, . . .

a) Write a rule for the nth term.

Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_1 = -8$, $d = -3$
$t_n = -8 - 3(n - 1)$
$t_n = -5 - 3n$

b) Use your rule to determine the 17th term.

For t_{17}, use $t_n = -5 - 3n$ and substitute: $n = 17$
$t_{17} = -5 - 3(17)$
$t_{17} = -56$

4. Use the given data about each arithmetic sequence to determine the indicated values.

a) $t_4 = -5$ and $t_7 = -20$; determine d and t_1

Use: $t_7 = t_4 + 3d$ Substitute: $t_7 = -20$, $t_4 = -5$
$-20 = -5 + 3d$
$-15 = 3d$
$d = -5$
Use: $t_1 = t_4 - 3d$ Substitute: $t_4 = -5$, $d = -5$
$t_1 = -5 - 3(-5)$
$t_1 = 10$

b) $t_1 = 3, d = 4$, and $t_n = 59$; determine n

Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_n = 59, t_1 = 3, d = 4$
$59 = 3 + 4(n - 1)$
$56 = 4n - 4$
$4n = 60$
$n = 15$

5. The steam clock in the Gastown district of Vancouver, B.C., displays the time on four faces and announces the quarter hours with a whistle chime that plays the tune *Westminster Quarters*. This sequence represents the number of tunes played from 1 to 3 days: 96, 192, 288, . . . Determine the number of tunes played in one year.

In one year, there are 365 days and 96(365), or 35 040 quarters. So, in one year, 35 040 tunes are played.

TEACHER NOTE

Achievement Indicators

Question 5 addresses AI 9.8: Solve a problem that involves an arithmetic sequence.

Question 7 addresses AI 9.1: Identify the assumption(s) made when defining an arithmetic series. AI 9.7: Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.

1.2

6. Multiple Choice For which series could you use $S_n = \frac{n(t_1 + t_n)}{2}$ to determine its sum?

A. $3 + 5 + 7 + 10 + 13 + 17 + 21$

B. $3 - 1 - 5 - 9 - 13 - 17 - 21$

C. $-3 - 5 - 8 - 10 - 13 - 15 - 18$

D. $3 - 1 + 5 - 3 + 7 - 5 + 9$

7. a) Create the first 5 terms of an arithmetic series with a common difference of -3.

Sample response: I chose a first term of 7.
$t_1 = 7$; so t_2 is $7 - 3 = 4$; t_3 is $4 - 3 = 1$; t_4 is $1 - 3 = -2$; and t_5 is $-2 - 3 = -5$
My arithmetic series is: $7 + 4 + 1 - 2 - 5 - \ldots$

b) Determine S_{26} for your series.

Sample response:

Use: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$ Substitute: $n = 26, t_1 = 7, d = -3$

$S_{26} = \frac{26[2(7) - 3(26 - 1)]}{2}$

$S_{26} = -793$

8. Determine the sum of this arithmetic series:

$-2 + 3 + 8 + 13 + \ldots + 158$

To determine n, use $t_n = t_1 + d(n - 1)$
Substitute: $t_n = 158$, $t_1 = -2$, $d = 5$
$158 = -2 + 5(n - 1)$
$160 = 5n - 5$
$165 = 5n$
$n = 33$
Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute: $n = 33$, $t_1 = -2$, $t_n = 158$
$S_{33} = \frac{33(-2 + 158)}{2}$
$S_{33} = 2574$

9. Use the given data about each arithmetic series to determine the indicated value.

a) $S_{17} = 106.25$ and $t_{17} = 8.25$; determine t_1

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ Substitute: $S_n = 106.25$, $n = 17$, $t_n = 8.25$
$106.25 = \frac{17(t_1 + 8.25)}{2}$
$212.5 = 17t_1 + 140.25$
$17t_1 = 72.25$
$t_1 = 4.25$

b) $S_{15} = 337.5$ and $t_1 = -2$; determine d

Use: $S_n = \frac{n(t_1 + t_n)}{2}$ to determine t_{15}.
Substitute: $S_n = 337.5$, $n = 15$, $t_1 = -2$
$337.5 = \frac{15(-2 + t_{15})}{2}$
$675 = -30 + 15t_{15}$
$705 = 15t_{15}$
$t_{15} = 47$
Use: $t_n = t_1 + d(n - 1)$ Substitute: $t_n = 47$, $t_1 = -2$, $n = 15$
$47 = -2 + d(15 - 1)$
$49 = 14d$
$d = 3.5$

TEACHER NOTE

Achievement Indicator

Questions 8 and 9 address AI 9.7: Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.

ANSWERS

1. A **3. a)** $t_n = -5 - 3n$ **b)** −56 **4. a)** −5; 10 **b)** 15 **5.** 35 040 **6.** B
8. 2574 **9. a)** 4.25 **b)** 3.5

1.3 Geometric Sequences

FOCUS **Solve problems involving geometric sequences.**

Get Started

For each sequence below, what are the next 2 terms? What is the rule?

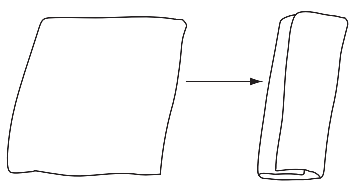
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
- $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
- $1, -3, 9, -27, 81, \ldots$

Sample response: For the first sequence, the denominators are the natural numbers.
The rule is: add 1 to the denominator; so the next two terms are: $\frac{1}{6}, \frac{1}{7}$
For the second sequence, the denominators are powers of 3.
The rule is: multiply the denominator by 3; so the next two terms are: $\frac{1}{243}, \frac{1}{729}$
For the third sequence, the rule is: multiply by -3; so the next two terms are: $-243, 729$

Construct Understanding

A French pastry called *mille feuille* or "thousand layers" is made from dough rolled into a square, buttered, and then folded into thirds to make three layers. This process is repeated many times. Each step of folding and rolling is called a turn.

How many turns are required to get more than 1000 layers?



Each turn produces 3 times as many layers, so start with 3 then keep multiplying by 3 until the number of layers is greater than 1000.

Number of turns	1	2	3	4	5	6	7
Number of layers	3	9	27	81	243	729	2187

From the table, 7 turns are required to produce more than 1000 layers.

Lesson Organizer
75 min

Key Math Concepts
In a geometric sequence, the ratio of any term to its preceding term is constant. We can use the common ratio and the first term to derive a rule for determining the *n*th term.

Curriculum Focus

SO	AI
RF10	10.1, 10.2, 10.3, 10.4, 10.9

Processes: PS, R, T

Student Materials
- scientific calculator
- graphing calculator (*optional*)
- Master 1.1b (*optional*)

Vocabulary
geometric sequence, common ratio, infinite geometric sequence, finite geometric sequence, divergent, convergent

TEACHER NOTE

DI: Extending Thinking

Have students record the last digit of each power of 3. Students use these digits to predict the last digit of 3^{10}, and to explain how a *mille feuille* cannot have exactly 1000 layers.
(*The pattern in the last digits is: 3, 9, 7, 1, 3, 9, 7, 1, . . . ; the last digit of 3^{10} is 9; since this is not 0, there cannot be exactly 1000 layers.*)

A **geometric sequence** is formed by multiplying each term after the 1st term by a constant, to determine the next term.

For example, $4, 4(3), 4(3)^2, 4(3)^3, \ldots$, is the geometric sequence:
$4, 12, 36, 108, \ldots$
The first term, t_1, is 4 and the constant is 3.
The constant is the **common ratio**, $\boldsymbol{r}$, of any term after the first, to the preceding term.
The common ratio is any non-zero real number.
To determine the common ratio, divide any term by the preceding term.
For the geometric sequence above:

THINK FURTHER

Why must r be non-zero?

If r were 0, all terms after the first term would be 0.

$r = \frac{12}{4}$ and $r = \frac{36}{12}$ and $r = \frac{108}{36}$

$r = 3$ $\quad$ $r = 3$ $\quad$ $r = 3$

The sequence $4, 12, 36, 108, \ldots$, is an **infinite geometric sequence** because it continues forever.
The sequence 4, 12, 36, 108 is a **finite geometric sequence** because the sequence is limited to a fixed number of terms.

Extra Material

Graphs to illustrate the three types of sequences described

Here are some other examples of geometric sequences.

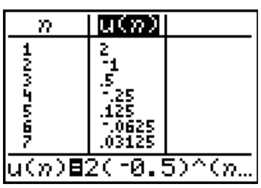
- This is an *increasing* geometric sequence because the terms are increasing: $2, 10, 50, 250, 1250, \ldots$
  The sequence is **divergent** because the terms do not approach a constant value.
- This is a geometric sequence that neither increases, nor decreases because consecutive terms have numerically greater values and different signs: $1, -2, 4, -8, 16, \ldots$
  The sequence is *divergent* because the terms do not approach a constant value.
- This is a *decreasing* geometric sequence because the terms are decreasing:
  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
  The sequence is **convergent** because the terms approach a constant value of 0.

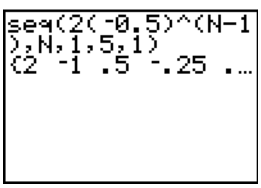
TECHNOLOGY NOTE

Graphing Calculator

Display the terms of a geometric sequence by defining a function, in sequence mode, using the general term of the sequence. Then use the table feature to display terms.

Or, use a sequence command to create a list of terms. For a sequence with $t_n = 2(-0.5)^{n-1}$:





Example 1 Determining a Term of a Given Geometric Sequence

a) Determine the 12th term of this geometric sequence:
512, −256, 128, −64, . . .

b) Identify the sequence as convergent or divergent.

SOLUTION

a) 512, −256, 128, −64, . . .

The common ratio is:

$$r = \frac{-256}{512}$$

$$r = -\frac{1}{2}$$

Multiply the first term, 512, by consecutive powers of $-\frac{1}{2}$.

$$t_1 = 512 \qquad t_2 = 512\left(-\frac{1}{2}\right) \qquad t_3 = 512\left(-\frac{1}{2}\right)^2 \qquad t_4 = 512\left(-\frac{1}{2}\right)^3$$

The exponent of each power of the common ratio is 1 less than the term number.
So, the 12th term is:

$$t_{12} = 512\left(-\frac{1}{2}\right)^{12-1}$$

$$t_{12} = 512\left(-\frac{1}{2}\right)^{11}$$

$$t_{12} = -\frac{1}{4}$$

The 12th term is $-\frac{1}{4}$.

b) Since consecutive terms approach a constant value of 0, the sequence is convergent.

Check Your Understanding

1. a) Determine the 10th term of this geometric sequence:
2, −6, 18, −54, . . .

b) Identify the sequence as convergent or divergent.

a) r is: $\frac{-6}{2} = -3$

$t_1 = 2$; $t_2 = 2(-3)$;

$t_3 = 2(-3)^2$; $t_4 = 2(-3)^3$

So, the 10th term is:

$t_{10} = 2(-3)^9$

$t_{10} = -39\ 366$

The 10th term is −39 366.

b) Since consecutive terms do not approach a constant value, the sequence is divergent.

THINK FURTHER

In *Example 1*, how can you predict whether a term is positive or negative?

The sign of the term depends on the sign of the power of $-\frac{1}{2}$. When this power has an even exponent, the term is positive. When this power has an odd exponent, the term is negative.

Check Your Understanding

Answers:

1. a) −39 366

b) divergent

Example 2 Creating a Geometric Sequence

Create a geometric sequence whose 5th term is 48.

SOLUTION

Work backward.

$t_5 = 48$

Choose a common ratio that is a factor of 48, such as 2.

Repeatedly divide 48 by 2.

$$t_5 = 48 \qquad t_4 = \frac{48}{2} \qquad t_3 = \frac{24}{2} \qquad t_2 = \frac{12}{2} \qquad t_1 = \frac{6}{2}$$

$$t_4 = 24 \qquad t_3 = 12 \qquad t_2 = 6 \qquad t_1 = 3$$

A possible geometric sequence is: 3, 6, 12, 24, 48, . . .

Check Your Understanding

2. Create a geometric sequence whose 6th term is 27.

$t_6 = 27$

Divide by a common ratio that is a factor of 27, such as 3.

$t_6 = 27$; $t_5 = \frac{27}{3}$, or 9;

$t_4 = \frac{9}{3}$, or 3; $t_3 = \frac{3}{3}$, or 1;

$t_2 = \frac{1}{3}$; $t_1 = \frac{\frac{1}{3}}{3}$, or $\frac{1}{9}$

A possible geometric sequence is:

$\frac{1}{9}, \frac{1}{3}$, 1, 3, 9, 27, . . .

THINK FURTHER

In *Example 2*, why does it make sense to choose a value for *r* that is a factor of 48? Could you choose any value for *r*?

When *r* is a factor of 48, I can use mental math to determine previous terms. No, *r* cannot be 0.

A geometric sequence with first term, t_1, and common ratio, r, can be written as:

$$\begin{array}{cccccccc} t_1, & t_1r, & t_1r^2, & t_1r^3, & t_1r^4, & \ldots, & t_1r^{n-1} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow \\ t_1 & t_2 & t_3 & t_4 & t_5 & & t_n \end{array}$$

The exponent of each power of the common ratio is 1 less than the term number.

The General Term of a Geometric Sequence

For a geometric sequence with first term, t_1, and common ratio, r, the general term, t_n, is:

$t_n = t_1r^{n-1}$

Recall that the product of two negative numbers is positive. So, a square number may be the product of two equal negative numbers or two equal positive numbers. For example,

when $r^2 = 4$

then $r = \pm\sqrt{4}$

and $r = 2$ or $r = -2$

Check Your Understanding

Answer:

2. $\frac{1}{9}, \frac{1}{3}$, 1, 3, 9, 27, . . .

To determine the common ratio of a geometric sequence, you may need to solve an equation of this form:

$$r^4 = 81$$

then $r^2 = 9$

and $r = 3$ or $r = -3$

THINK FURTHER

Why is $r^2 \neq -9$?

The square of a real number cannot be negative.

Example 3 Determining Terms and the Number of Terms in a Finite Geometric Sequence

In a finite geometric sequence, $t_1 = 5$ and $t_5 = 1280$

a) Determine t_2 and t_6.

b) The last term of the sequence is 20 480. How many terms are in the sequence?

SOLUTION

a) Determine the common ratio.

Use: $t_n = t_1r^{n-1}$ Substitute: $n = 5, t_5 = 1280, t_1 = 5$

$1280 = 5r^{5-1}$ Simplify.

$1280 = 5r^4$ Divide each side by 5.

$256 = r^4$ Take the fourth root of each side.

$\pm\sqrt[4]{256} = r$

$r = -4$ or $r = 4$

There are 2 possible values for r.

When $r = -4$, then t_2 is $5(-4) = -20$

When $r = 4$, then t_2 is $5(4) = 20$

To determine t_6, use: $t_n = t_1r^{n-1}$

Substitute: $n = 6, t_1 = 5, r = -4$

$t_6 = 5(-4)^{6-1}$

$t_6 = 5(-4)^5$

$t_6 = -5120$

Substitute: $n = 6, t_1 = 5, r = 4$

$t_6 = 5(4)^{6-1}$

$t_6 = 5(4)^5$

$t_6 = 5120$

So, t_2 is -20 or 20, and t_6 is -5120 or 5120.

b) Since the last term is positive, use the positive value of r.

$t_n = t_1r^{n-1}$ Substitute: $t_n = 20\,480, t_1 = 5, r = 4$

$20\,480 = 5(4)^{n-1}$ Divide each side by 5.

$4096 = 4^{n-1}$

Use guess and test to determine which power of 4 is equal to 4096.

Guess: $4^4 = 256$ This is too low.

Guess: $4^6 = 4096$ This is correct.

So, $4^6 = 4^{n-1}$ Equate exponents.

$6 = n - 1$

$n = 7$

There are 7 terms in the sequence.

Animation

Check Your Understanding

3. In a finite geometric sequence, $t_1 = 7$ and $t_5 = 567$

a) Determine t_2 and t_6.

b) The last term is 45 927. How many terms are in the sequence?

a) **Use: $t_n = t_1r^{n-1}$**

Substitute: $n = 5, t_5 = 567, t_1 = 7$

$567 = 7r^{5-1}$

$81 = r^4$

$r = -3$ or $r = 3$

When $r = -3$, then t_2 is $7(-3) = -21$

For t_6, use: $t_n = t_1r^{n-1}$

Substitute: $n = 6, t_1 = 7$, and $r = -3$

$t_6 = 7(-3)^{6-1}$

$t_6 = -1701$

When $r = 3$, then t_2 is $7(3) = 21$

Use: $t_n = t_1r^{n-1}$

Substitute: $n = 6, t_1 = 7$, and $r = 3$

$t_6 = 7(3)^{6-1}$

$t_6 = 1701$

b) **Use the positive value of r in $t_n = t_1r^{n-1}$.**

Substitute: $t_n = 45\,927, t_1 = 7, r = 3$

$45\,927 = 7(3)^{n-1}$, or $6561 = 3^{n-1}$

$3^8 = 6561$

So, $3^8 = 3^{n-1}$

$8 = n - 1$

$n = 9$

There are 9 terms.

Check Your Understanding

Answers:

3. **a)** -21 or 21, -1701 or 1701

b) 9 terms

Example 4 Using a Geometric Sequence to Solve a Problem

The population of Airdrie, Alberta, experienced an average annual growth rate of about 9% from 2001 to 2006. The population in 2006 was 28 927. Estimate the population in each year to the nearest thousand.

a) 2011

b) 2030, the 125th anniversary of Alberta becoming part of Canada

What assumption did you make? Is this assumption reasonable?

SOLUTION

a) A growth rate of 9% means that each year the population increases by 9%, or 0.09.

The population in 2006 was 28 927.

So, the population in 2007 was:

$28\,927 + 9\%$ of $28\,927$

$= 28\,927 + 0.09(28\,927)$ Remove 28 927 as a common factor.

$= 28\,927(1 + 0.09)$

$= 28\,927(1.09)$

Increasing a quantity by 9% is the same as multiplying it by 1.09. So, to determine a population with a growth rate of 9%, multiply the current population by 1.09.

The annual populations form a geometric sequence with 1st term 28 927 and common ratio 1.09.

The population in 2006 is the 1st term. So, the population in 2011 is the 6th term: $28\,927(1.09)^5 = 44\,507.7751...$

The population in 2011 is approximately 45 000.

b) To predict the population in 2030, determine n, the number of years from 2006 to 2030:

$n = 2030 - 2006$

$n = 24$

The population in 2030 is: $28\,927(1.09)^{24} = 228\,843.903...$

The population in 2030 will be approximately 229 000.

We assume that the population increase of 9% annually continues. This assumption may be false because the rate of growth may change in future years. This assumption is reasonable for a short time span, but not for a longer time span, such as 100 years.

Check Your Understanding

4. Statistics Canada estimates the population growth of Canadian cities, provinces, and territories. The population of Nunavut is expected to grow annually by 0.8%. In 2009, its population was about 30 000. Estimate the population in each year to the nearest thousand.

a) 2013

b) 2049; Nunavut's 50th birthday

a) For a growth rate of 0.8%, multiply the current population by 1.008.
The annual populations form a geometric sequence with 1st term 30 000 and common ratio 1.008.
The population in 2009 is the 1st term. So, the population in 2013 is the 5th term:

$30\,000(1.008)^4 = 30\,971.581...$

The population in 2013 will be approximately 31 000.

b) Determine n, the number of years from 2009 to 2049:

$n = 2049 - 2009$

$n = 40$

The population in 2049 is:
$30\,000(1.008)^{40} = 41\,261.265...$
The population in 2049 will be approximately 41 000.

Check Your Understanding

Answers:

4. a) approximately 31 000

b) approximately 41 000

Discuss the Ideas

1. How do you determine whether a given sequence is geometric? What assumptions do you make?

After the first term, I divide each term by its preceding term. If these quotients are equal, then the sequence is geometric and the quotient is the common ratio. I assume that the pattern in the terms continues.

TEACHER NOTE

Achievement Indicator

Question 1 addresses AI 10.1: Identify assumptions made when identifying a geometric sequence.

2. Which geometric sequences are created when $r = 1$? $r = -1$?

When $r = 1$, all terms in the sequence are equal; for example, 3, 3, 3, 3, . . .
When $r = -1$, the terms have the same numerical value, but alternate in sign; for example, $-4, 4, -4, 4, \ldots$

Exercises

A

3. Which sequences could be geometric? If a sequence is geometric, state its common ratio.

a) 1, 2, 4, 8, 16, . . .

The sequence is geometric.
r **is:** $\frac{2}{1} = 2$

b) 4, 9, 16, 25, 36, . . .

The sequence is not geometric.

c) -3, 2, 7, 12, 17, . . .

The sequence is not geometric.

d) 6, 0.6, 0.06, 0.006, . . .

The sequence is geometric.
r **is:** $\frac{0.6}{6} = 0.1$

e) 10, 100, 1000, 10 000

The sequence is geometric.
r **is:** $\frac{100}{10} = 10$

f) 2, 4, 6, 8, 10, . . .

The sequence is not geometric.

4. State the common ratio, then write the next 3 terms of each geometric sequence.

a) $-1, -3, -9, \ldots$

r is $\frac{-3}{-1} = 3$
The next 3 terms are:
$-27, -81, -243$

b) $48, 24, 12, \ldots$

r is $\frac{24}{48} = 0.5$
The next 3 terms are:
$6, 3, 1.5$

c) $4, -2, 1, \ldots$

r is $\frac{-2}{4} = -0.5$
The next 3 terms are:
$-0.5, 0.25, -0.125$

d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \ldots$

r is $\frac{1}{6} \div \frac{1}{2} = \frac{1}{3}$
The next 3 terms are:
$\frac{1}{54}, \frac{1}{162}, \frac{1}{486}$

5. For each geometric sequence, determine the indicated value.

a) $3, 6, 12, \ldots$; determine t_7

r is: $\frac{6}{3} = 2$
Use: $t_n = t_1r^{n-1}$
Substitute: $n = 7$, $t_1 = 3$, $r = 2$
$t_7 = 3(2)^{7-1}$
$t_7 = 192$

b) $18, 9, 4.5, \ldots$; determine t_6

r is: $\frac{9}{18} = 0.5$
Use: $t_n = t_1r^{n-1}$
Substitute: $n = 6$, $t_1 = 18$, $r = 0.5$
$t_6 = 18(0.5)^{6-1}$
$t_6 = 0.5625$

c) $23, -46, 92, \ldots$; determine t_{10}

r is: $\frac{-46}{23} = -2$
Use: $t_n = t_1r^{n-1}$
Substitute:
$n = 10$, $t_1 = 23$, $r = -2$
$t_{10} = 23(-2)^{10-1}$
$t_{10} = -11\,776$

d) $2, \frac{1}{2}, \frac{1}{8}, \ldots$; determine t_5

r is: $\frac{\frac{1}{2}}{2} = \frac{1}{4}$
Use: $t_n = t_1r^{n-1}$
Substitute:
$n = 5$, $t_1 = 2$, $r = \frac{1}{4}$
$t_5 = 2\left(\frac{1}{4}\right)^{5-1}$
$t_5 = \frac{1}{128}$

TEACHER NOTE

DI: Common Difficulties

For students having difficulty determining unknown terms in a geometric sequence, suggest a diagrammatic approach. For example, suppose t_5 and r are known; use this diagram to determine t_9:

$$t_5 \xrightarrow{\times r} t_6 \xrightarrow{\times r} t_7 \xrightarrow{\times r} t_8 \xrightarrow{\times r} t_9$$

$t_9 = t_5r^4$

B

6. Write the first 4 terms of each geometric sequence, given the 1st term and the common ratio. Identify the sequence as decreasing, increasing, or neither. Justify your answers.

a) $t_1 = -3$; $r = 4$

$t_1 = -3$
t_2 is $(-3)(4) = -12$
t_3 is $(-12)(4) = -48$
t_4 is $(-48)(4) = -192$
The sequence is decreasing because the terms are decreasing.

b) $t_1 = 5$; $r = 2$

$t_1 = 5$
t_2 is $(5)(2) = 10$
t_3 is $(10)(2) = 20$
t_4 is $(20)(2) = 40$
The sequence is increasing because the terms are increasing.

c) $t_1 = -0.5; r = -3$

$t_1 = -0.5$
t_2 is $(-0.5)(-3) = 1.5$
t_3 is $(1.5)(-3) = -4.5$
t_4 is $(-4.5)(-3) = 13.5$
The sequence neither increases nor decreases because the terms alternate signs.

d) $t_1 = \frac{1}{2}; r = \frac{2}{3}$

$t_1 = \frac{1}{2}$
t_2 is $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$
t_3 is $\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{9}$
t_4 is $\left(\frac{2}{9}\right)\left(\frac{2}{3}\right) = \frac{4}{27}$
The sequence is decreasing because the terms are decreasing.

7. Write the first 5 terms of each geometric sequence.

a) the 6th term is 64

Sample response: $t_6 = 64$
Divide by a common ratio that is a factor of 64, such as -2.

$t_6 = 64$ $\quad t_5 = \frac{64}{-2}$
$t_5 = -32$
$t_4 = \frac{-32}{-2}$ $\quad t_3 = \frac{16}{-2}$
$t_4 = 16$ $\quad t_3 = -8$
$t_2 = \frac{-8}{-2}$ $\quad t_1 = \frac{4}{-2}$
$t_2 = 4$ $\quad t_1 = -2$

A possible geometric sequence is:
$-2, 4, -8, 16, -32, \ldots$

b) the 1st term is $\frac{3}{4}$

Sample response:
Choose a value for r, such as $r = 4$.

$t_1 = \frac{3}{4}$ $\quad t_2 = \left(\frac{3}{4}\right)(4)$
$t_2 = 3$
$t_3 = (3)(4)$ $\quad t_4 = (12)(4)$
$t_3 = 12$ $\quad t_4 = 48$
$t_5 = (48)(4)$
$t_5 = 192$

A possible geometric sequence is:
$\frac{3}{4}, 3, 12, 48, 192, \ldots$

c) every term is negative

Sample response: Choose a negative 1st term and a positive common ratio, such as $t_1 = -5$ and $r = 2$.

$t_1 = -5$ $\quad t_2 = (-5)(2)$
$t_2 = -10$
$t_3 = (-10)(2)$ $\quad t_4 = (-20)(2)$
$t_3 = -20$ $\quad t_4 = -40$
$t_5 = (-40)(2)$
$t_5 = -80$

A possible geometric sequence is:
$-5, -10, -20, -40, -80, \ldots$

d) every term is an even number

Sample response: Choose an even 1st term and an odd or even common ratio, such as $t_1 = 4$ and $r = 3$

$t_1 = 4$ $\quad t_2 = (4)(3)$
$t_2 = 12$
$t_3 = (12)(3)$ $\quad t_4 = (36)(3)$
$t_3 = 36$ $\quad t_4 = 108$
$t_5 = (108)(3)$
$t_5 = 324$

A possible geometric sequence is:
$4, 12, 36, 108, 324, \ldots$

TEACHER NOTE

Achievement Indicator

Questions 6 and 7 address AI 10.2: Provide and justify an example of a geometric sequence.

8. Use the given data about each finite geometric sequence to determine the indicated values.

a) Given $t_1 = -1$ and $r = -2$

i) Determine t_9.

Use: $t_n = t_1r^{n-1}$ Substitute: $n = 9$, $t_1 = -1$, $r = -2$
$t_9 = (-1)(-2)^{9-1}$
$t_9 = (-1)(-2)^8$
$t_9 = -256$

ii) The last term is −4096. How many terms are in the sequence?

Use $t_n = t_1r^{n-1}$ to determine n.
Substitute: $t_n = -4096$, $t_1 = -1$, $r = -2$
$-4096 = (-1)(-2)^{n-1}$ Divide by -1.
$4096 = (-2)^{n-1}$
$(-2)^{12} = (-2)^{n-1}$ Equate exponents.
$12 = n - 1$
$n = 13$
There are 13 terms in the sequence.

b) Given $t_1 = 0.002$ and $t_4 = 2$

i) Determine t_7.

Use $t_n = t_1r^{n-1}$ to determine the common ratio, r.
Substitute: $n = 4$, $t_4 = 2$, $t_1 = 0.002$
$2 = 0.002r^{4-1}$
$2 = 0.002r^3$ Divide each side by 0.002.
$1000 = r^3$
$\sqrt[3]{1000} = r$
$r = 10$
To determine t_7, use: $t_n = t_1r^{n-1}$
Substitute: $n = 7$, $t_1 = 0.002$, and $r = 10$
$t_7 = 0.002(10)^{7-1}$
$t_7 = 0.002(10)^6$
$t_7 = 2000$

ii) Determine which term has the value 20 000.

Use $t_n = t_1r^{n-1}$ to determine n.
Substitute: $t_n = 20\,000$, $t_1 = 0.002$, $r = 10$
$20\,000 = 0.002(10)^{n-1}$
$10\,000\,000 = 10^{n-1}$
$10^7 = 10^{n-1}$
$7 = n - 1$
$n = 8$
20 000 is the 8th term.

TEACHER NOTE

Achievement Indicator

Question 8 addresses AI 10.4: Determine t_1, r, n, or t_n in a problem that involves a geometric sequence.

9. a) Insert three numbers between 8 and 128, so the five numbers form an arithmetic sequence. Explain what you did.

The sequence has the form: 8, $8 + d$, $8 + 2d$, $8 + 3d$, 128
Write $128 = 8 + 4d$, then solve for d to get $d = 30$.
The arithmetic sequence is: 8, 38, 68, 98, 128

b) Insert three numbers between 8 and 128, so the five numbers form a geometric sequence. Explain what you did.

The sequence has the form: 8, $8r$, $8r^2$, $8r^3$, 128
Write $128 = 8r^4$, then solve for r to get $r^4 = 16$, so $r = 2$ or -2.
The geometric sequence is: 8, 16, 32, 64, 128;
or 8, −16, 32, −64, 128

c) What was similar about your strategies in parts a and b? What was different?

For each sequence, I wrote an equation for the 5th term, then solved the equation to determine the common difference and common ratio. For the arithmetic sequence, I added the common difference to get the next terms. For the geometric sequences, there were two possible common ratios, and I multiplied by each common ratio to get the next terms.

10. Suppose a person is given 1¢ on the first day of April; 3¢ on the second day; 9¢ on the third day, and so on. This pattern continues throughout April.

a) About how much money will the person be given on the last day of April?

There are 30 days in April.
The daily amounts, in cents, form this geometric sequence:
$1, 1(3), 1(3)^2, \ldots, 1(3)^{29}$
The amount on the last day, in cents, is $1(3)^{29} \doteq 6.863 \times 10^{13}$
Divide by 100 to convert the amount to dollars:
approximately $\$6.863 \times 10^{11}$

b) Why might it be difficult to determine the exact amount using a calculator?

A calculator screen shows only 10 digits, and the number of digits in the amount of money in dollars is greater than 10.

11. In a geometric sequence, $t_3 = 9$ and $t_6 = 1.125$; determine t_7 and t_9.

Use $t_n = t_1 r^{n-1}$ twice to get two equations.
For t_3, substitute: $n = 3$, $t_n = 9$
$9 = t_1 r^{3-1}$
$9 = t_1 r^2$ ①
For t_6, substitute: $n = 6$, $t_n = 1.125$
$1.125 = t_1 r^{6-1}$
$1.125 = t_1 r^5$ ②
Write equation ② as: $1.125 = t_1 r^2(r^3)$
From equation ①, substitute $t_1 r^2 = 9$
$1.125 = 9r^3$ Divide by 9.
$0.125 = r^3$
$r = \sqrt[3]{0.125}$
$r = 0.5$
So, $t_7 = t_6 r$ and, $t_9 = t_7 r^2$
$= 1.125(0.5)$ $= 0.5625(0.5)^2$
$= 0.5625$ $= 0.140\ 625$

12. An arithmetic sequence and a geometric sequence have the same first term. The common difference and common ratio are equal and greater than 1. Which sequence increases more rapidly as more terms are included? Use examples to explain.

This arithmetic sequence has $t_1 = 3$ and $d = 4$:
3, 7, 11, 15, 19, 23, . . .
This geometric sequence has $t_1 = 3$ and $r = 4$:
3, 12, 48, 192, 768, 3072, . . .
The geometric sequence increases more rapidly because we are multiplying instead of adding to get the next term.

13. A ream of paper is about 2 in. thick. Imagine a ream of paper that is continually cut in half and the two halves stacked one on top of the other. How many cuts have to be made before the stack of paper is taller than 318 ft., the height of Le Chateau York in Winnipeg, Manitoba?

TEACHER NOTE

Achievement Indicator
Questions 13 and 14 address AI 10.9:
Solve a problem that involves a geometric sequence.

Let the number of cuts be n.
The heights of the stacks of paper form, in inches, a geometric sequence with 1st term 2 and common ratio 2:
$2,\quad 2(2),\quad 2(2)^2,\quad 2(2)^3,\quad \ldots\quad 2(2)^n$
Write 318 ft. in inches: 318(12 in.) = 3816 in.
Write an equation:
$2(2)^n = 3816$ Solve for n.
$2^n = 1908$
Use guess and test: $2^{10} = 1024$; $2^{11} = 2048$
10 cuts will not be enough.
11 cuts will produce a stack that is: $2(2)^{11}$ in. = 4096 in. high
11 cuts have to be made.

14. Between the Canadian censuses in 2001 and 2006, the number of people who could converse in Cree had increased by 7%. In 2006, 87 285 people could converse in Cree. Assume the 5-year increase continues to be 7%. Estimate to the nearest hundred how many people will be able to converse in Cree in 2031.

To model a growth rate of 7%, multiply by 1.07.
The number of people every 5 years form a geometric sequence with first term 87 285 and common ratio 1.07.
Every 5 years is: 2006, 2011, 2016, 2021, 2026, 2031, . . .
So, the number of people in 2031 is:
$87\ 285(1.07)^5 = 122\ 421.7278...$
The number of people who will be able to converse in Cree in 2031 will be approximately 122 400.

C

15. A farmer in Saskatchewan wants to estimate the value of a new combine after several years of use. A new combine worth \$370 000 depreciates in value by about 10% each year.

a) Estimate the value of the combine at the end of each of the first 5 years. Write the values as a sequence.

When the value decreases by 10%, the new value is 90% of the original value.
To determine a depreciation value of 10%, multiply by 0.9.
The values, in dollars, at the end of each of the first 5 years are:
$370\ 000(0.9)$, $370\ 000(0.9)^2$, $370\ 000(0.9)^3$, $370\ 000(0.9)^4$, $370\ 000(0.9)^5$
The values, to the nearest dollar, are: \$333 000, \$299 700, \$269 730, \$242 757, \$218 481

b) What type of sequence did you write in part a? Explain your reasoning.

The sequence is geometric because I multiplied by a constant to get each value from the preceding value.

c) Predict the value of the combine at the end of 10 years.

At the end of 10 years, to the nearest dollar, the value is:
$\$370\ 000(0.9)^{10} = \$129\ 011$

16. **a)** Show that squaring each term in a geometric sequence produces the same type of sequence. What is the common ratio?

Consider the sequence: $t_1, t_1r, t_1r^2, t_1r^3, t_1r^4, \ldots, t_1r^{n-1}$
Square each term. The new sequence is:
$t_1^2, t_1^2r^2, t_1^2r^4, t_1^2r^6, t_1^2r^8, \ldots, t_1^2r^{2n-2}$
This is a geometric sequence with 1st term t_1^2 and common ratio r^2.

b) Show that raising each term in a geometric sequence to the mth power of each term produces the same type of sequence. What is the common ratio?

Consider the sequence: $t_1, t_1r, t_1r^2, t_1r^3, \ldots, t_1r^{n-1}$
Raise each term to the mth power.
The new sequence is: $t_1^m, t_1^m r^m, t_1^m r^{2m}, t_1^m r^{3m}, \ldots, t_1^m r^{mn-m}$
This is a geometric sequence with 1st term t_1^m and common ratio r^m.

Multiple-Choice Questions

1. Which expression below represents the nth term of this geometric sequence?

$9, -6, 4, -\frac{8}{3}, \ldots$

A. $9\left(\frac{2}{3}\right)^{n-1}$ **B.** $9\left(-\frac{2}{3}\right)^{n-1}$ **C.** $\frac{2}{3}(9^{n-1})$ **D.** $-\frac{2}{3}(9^{n-1})$

2. Which geometric sequence does not have a common ratio of −0.5?

A. −5, 2.5, −1.25, 0.625, . . . **B.** 6, −3, 1.5, −0.75, . . .

C. $\frac{1}{200}, -\frac{1}{100}, \frac{1}{50}, -\frac{1}{25}, \ldots$ **D.** $-\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, \ldots$

3. The value of a car in each of its first 3 years is: \$24 000, \$20 400, \$17 340
These amounts form a sequence.
Which statement describes this sequence?

A. arithmetic with common difference \$3600

B. geometric with common ratio approximately 1.18

C. geometric with common ratio 0.85

D. neither arithmetic nor geometric

TEACHER NOTE

Solution strategy: Calculate the difference and quotient of each pair of consecutive terms to see which values are constant.

Study Note

Can a sequence be both arithmetic and geometric? Explain.

A sequence can be both arithmetic and geometric when all the terms are equal. For example: 3, 3, 3, 3, 3, . . . is an arithmetic sequence with 1st term 3 and common difference 0; and 3, 3, 3, 3, 3, . . . is a geometric sequence with 1st term 3 and common ratio 1.

ANSWERS

3. a) 2 **d)** 0.1 **e)** 10 **4. a)** 3; −27, −81, −243 **b)** 0.5; 6, 3, 1.5
c) −0.5; −0.5, 0.25, −0.125 **d)** $\frac{1}{3}; \frac{1}{54}, \frac{1}{162}, \frac{1}{486}$ **5. a)** 192 **b)** 0.5625 **c)** −11 776
d) $\frac{1}{128}$ **6. a)** −3, −12, −48, −192; decreasing **b)** 5, 10, 20, 40; increasing
c) −0.5, 1.5, −4.5, 13.5; neither **d)** $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}$; decreasing
8. a) i) −256 **ii)** 13 **b) i)** 2000 **ii)** t_8 **9. a)** 8, 38, 68, 98, 128, . . .
b) 8, −16, 32, −64, 128, . . . or 8, 16, 32, 64, 128, . . . **10. a)** approximately \$6.863 × 10^{11}
11. 0.5625; 0.140 625 **12.** geometric **13.** 11 cuts **14.** approximately 122 400 people
15. a) \$333 000, \$299 700, \$269 730, \$242 757, \$218 481 **b)** geometric **c)** \$129 011

Multiple Choice
1. B **2.** C **3.** C

1.4 Geometric Series

FOCUS **Derive a rule to determine the sum of *n* terms of a geometric series, then solve related problems.**

Get Started

Two geometric sequences have the same first term but the common ratios are opposite integers. Which corresponding terms are equal? Which corresponding terms are different? Use an example to explain.

Suppose the 1st term is 3 and the common ratios are 2 and −2.

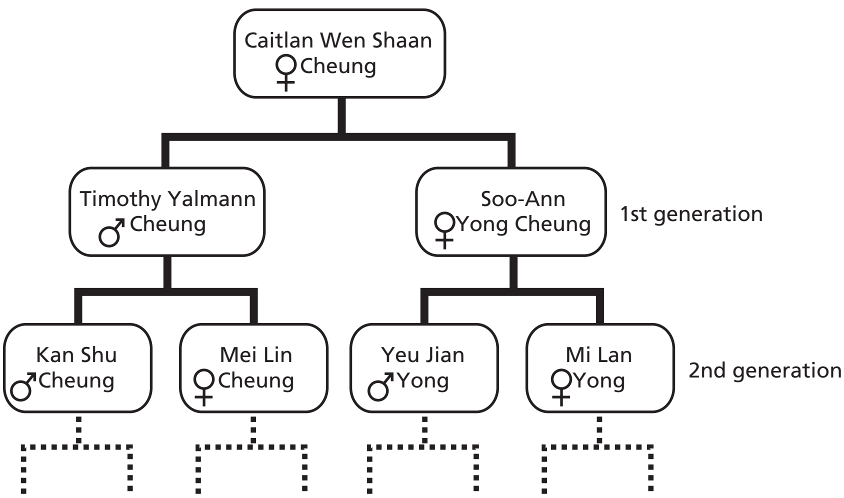
One sequence is: 3, 3(2), $3(2)^2$, $3(2)^3$, $3(2)^4$, . . .; or 3, 6, 12, 24, 48, . . .

The other sequence is: 3, 3(−2), $3(-2)^2$, $3(-2)^3$, $3(-2)^4$, . . .; or 3, −6, 12, −24, 48, . . .

The odd numbered terms are equal and the even numbered terms are different.

Construct Understanding

Caitlan traced her direct ancestors, beginning with her 2 parents, 4 grandparents, 8 great-grandparents, and so on.



Determine the total number of Caitlan's direct ancestors in 20 generations.

The numbers in each generation form a geometric sequence with common ratio 2: 2, 4, 8, 16, . . .; or 2^1, 2^2, 2^3, 2^4, . . .

Make a table for the total number of direct ancestors and look for patterns.

Lesson Organizer
75 min

Key Math Concepts
A geometric series is the sum of the terms in a geometric sequence.

Curriculum Focus

SO	AI
RF10	10.1, 10.5, 10.6, 10.9

Processes: PS, R, T

Student Materials
- scientific calculator
- graphing calculator (*optional*)

Vocabulary
geometric series

TEACHER NOTE

DI: Common Difficulties

For students who have difficulty organizing their solutions for this problem, suggest a 3-column table to track: Generation; Number of Ancestors; Total Number of Ancestors.

Generation	Number of Ancestors	Total Ancestors
1	2	2
2	4	6

TEACHER NOTE

DI: Extending Thinking

Ask students:

- What if Caitlan were included in the total number of people? What totals would you see in the first 4 generations? *(1, 3, 7, 15)*
- What rule can you write to determine the total number of people, including Caitlan, for n generations? *($2^n - 1$)*
- How many generations would Caitlin have to list to show about 1 million direct ancestors? *(20, since $2^{20} - 1 = 1\ 048\ 575$)*

Generation	Total number of direct ancestors	Generation	Total number of direct ancestors
1	2	5	30 + 32 = 62
2	2 + 4 = 6	6	62 + 64 = 126
3	6 + 8 = 14	7	126 + 128 = 254
4	14 + 16 = 30	8	254 + 256 = 510

Each number is a term in the geometric sequence minus 2:

In 1 generation: $4 - 2 = 2$ or $2^2 - 2 = 2$

In 2 generations: $8 - 2 = 6$ or $2^3 - 2 = 6$

In 3 generations: $16 - 2 = 14$ or $2^4 - 2 = 14$, and so on

So, the total number of direct ancestors in 20 generations is:

$2^{21} - 2 = 2\ 097\ 150$

THINK FURTHER

How could you determine S_n when $r = 1$?

S_n is the product of the first term and n.

A **geometric series** is the sum of the terms of a geometric sequence.
For example, a geometric sequence is: 6, 12, 24, 48, . . .
The related geometric series is: 6 + 12 + 24 + 48 + . . .
S_n represents the partial sum of n terms of the series; for example,

$S_1 = t_1$ | $S_2 = t_1 + t_2$ | $S_3 = t_1 + t_2 + t_3$
$S_1 = 6$ | $S_2 = 6 + 12$ | $S_3 = 6 + 12 + 24$
 | $S_2 = 18$ | $S_3 = 42$

Animation

Extra Material

Numerical example to support the general derivation of the formula given here

The nth term of a geometric series is the nth term of the related geometric sequence.
To derive a rule for determining the sum of the first n terms in any geometric series: $t_1 + t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \ldots + t_1r^{n-1}$
write the sum, multiply each side by the common ratio, r, write the new sum, then subtract vertically.

$$\begin{aligned}
S_n &= t_1 + t_1r + t_1r^2 + t_1r^3 + \ldots + t_1r^{n-2} + t_1r^{n-1} \\
-(rS_n &= \qquad t_1r + t_1r^2 + t_1r^3 + t_1r^4 + \ldots + t_1r^{n-1} + t_1r^n) \\
S_n - rS_n &= t_1 - t_1r^n \qquad \text{Factor each side.} \\
S_n(1 - r) &= t_1(1 - r^n) \qquad \text{Divide each side by } (1 - r) \text{ to solve for } S_n. \\
S_n &= \frac{t_1(1 - r^n)}{1 - r}, r \neq 1
\end{aligned}$$

THINK FURTHER

Why is 1 not a permissible value for r?

When $r = 1$, the denominator of $\frac{t_1(1 - r^n)}{1 - r}$ is 0 and a fraction with denominator 0 is undefined.

The Sum of n Terms of a Geometric Series

For the geometric series $t_1 + t_1r + t_1r^2 + \ldots + t_1r^{n-1}$,

the sum of n terms, S_n, is: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

THINK FURTHER

Why can the rule also be written as $S_n = \frac{t_1(r^n - 1)}{r - 1}, r \neq 1$?
When does it make sense to use this form of the rule?

Multiplying a fraction by 1 does not change its value. So, multiply $\frac{t_1(1 - r^n)}{1 - r}$ by 1, written as $\frac{-1}{-1}$, to get $\frac{t_1(r^n - 1)}{r - 1}$. Use this form when r is greater than 1 or less than −1.

Example 1 Determining the Sum of Given Terms of a Geometric Series

Determine the sum of the first 15 terms of this geometric series:
$40 - 20 + 10 - 5 + 2.5 - \ldots$
Write the sum to the nearest hundredth.

SOLUTION

$40 - 20 + 10 - 5 + 2.5 - \ldots$

$t_1 = 40$ and r is $\frac{-20}{40} = -\frac{1}{2}$

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 15, t_1 = 40, r = -\frac{1}{2}$

$$S_{15} = \frac{40\left(1 - \left(-\frac{1}{2}\right)^{15}\right)}{1 - \left(-\frac{1}{2}\right)}$$ Use a calculator.

$S_{15} = 26.6674\ldots$

The sum of the first 15 terms is approximately 26.67.

Check Your Understanding

1. Determine the sum of the first 12 terms of this geometric series:
$3 + 12 + 48 + 192 + \ldots$

$t_1 = 3$ and r is: $\frac{12}{3} = 4$

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 12, t_1 = 3, r = 4$

$S_{12} = \frac{3(1 - 4^{12})}{1 - 4}$

$S_{12} = 16\,777\,215$

The sum of the first 12 terms is 16 777 215.

Example 2 Determining Terms of a Geometric Series

The sum of the first 10 terms of a geometric series is −29 524. The common ratio is −3. Determine the 1st term.

SOLUTION

Suppose the geometric series has 1st term, t_1, and common ratio, r.

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 10, S_n = -29\,524, r = -3$

$$-29\,524 = \frac{t_1(1 - (-3)^{10})}{1 - (-3)}$$

Check Your Understanding

2. The sum of the first 14 terms of a geometric series is 16 383. The common ratio is −2. Determine the 1st term.

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 14, S_n = 16\,383, r = -2$

Check Your Understanding

Answers:

1. 16 777 215 **2.** −3

$$16\,383 = \frac{t_1(1 - (-2)^{14})}{1 - (-2)}$$
$$16\,383 = \frac{t_1(1 - 16\,384)}{3}$$
$$16\,383 = -5461t_1$$
$$\frac{16\,383}{-5461} = t_1$$
$$-3 = t_1$$
The 1st term is -3.

$-29\,524 = \dfrac{t_1(1 - 59\,049)}{4}$	Simplify.
$-29\,524 = -14\,762t_1$	Solve for t_1.
$\dfrac{-29\,524}{-14\,762} = t_1$	
$2 = t_1$	

The 1st term is 2.

THINK FURTHER

In *Example 2*, how can you check the answer?

I can substitute $n = 10$, $t_1 = 2$, and $r = -3$ in $S_n = \dfrac{t_1(1 - r^n)}{1 - r}$, and check that I get a sum of $-29\,524$.

Example 3 Determining the Sum of a Geometric Series

Calculate the sum of this geometric series:
$6 + 12 + 24 + 48 + \ldots + 12\,288$

SOLUTION

$6 + 12 + 24 + 48 + \ldots + 12\,288$

$t_1 = 6$ and r is $\dfrac{12}{6} = 2$

Determine n, the number of terms in the series.

Use: $t_n = t_1r^{n-1}$	Substitute: $t_n = 12\,288$, $t_1 = 6$, $r = 2$
$12\,288 = 6(2)^{n-1}$	Simplify. Divide each side by 6.
$2048 = 2^{n-1}$	Use guess and test to write 2048 as a power of 2.
$2^{11} = 2^{n-1}$	Equate exponents.
$11 = n - 1$	
$n = 12$	

There are 12 terms in the series.

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 12$, $t_1 = 6$, $r = 2$

$$S_{12} = \frac{6(1 - 2^{12})}{1 - 2}$$
$$S_{12} = 24\,570$$

The sum is 24 570.

Check Your Understanding

3. Calculate the sum of this geometric series:
$-3 - 15 - 75 - \ldots - 46\,875$

$t_1 = -3$ and r is $\dfrac{-15}{-3} = 5$

To determine n, use: $t_n = t_1r^{n-1}$

Substitute: $t_n = -46\,875$, $t_1 = -3$, $r = 5$

$$-46\,875 = (-3)(5)^{n-1}$$
$$15\,625 = 5^{n-1}$$
$$5^6 = 5^{n-1}$$
$$6 = n - 1$$
$$n = 7$$

There are 7 terms in the series.

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 7$, $t_1 = -3$, $r = 5$

$$S_7 = \frac{(-3)(1 - 5^7)}{1 - 5}$$
$$S_7 = -58\,593$$

The sum is $-58\,593$.

Check Your Understanding

Answer:

3. $-58\,593$

Example 4 Using a Geometric Series to Model and Solve a Problem

A person takes tablets to cure an ear infection. Each tablet contains 200 mg of an antibiotic. About 12% of the mass of the antibiotic remains in the body when the next tablet is taken. Determine the mass of antibiotic in the body after each number of tablets has been taken.

a) 3 tablets **b)** 12 tablets

SOLUTION

a) Determine the mass of the antibiotic in the body for 1 to 3 tablets.

Number of tablets	Mass of antibiotic (mg)
1	200
2	$200 + 200(0.12) = 224$
3	$200 + 200(0.12) + 200(0.12)^2 = 226.88$

The problem can be modelled by a geometric series, which has 3 terms because 3 tablets were taken:
$200 + 200(0.12) + 200(0.12)^2$
The sum is 226.88. So, after taking the 3rd tablet, the total mass of antibiotic in the person's body is 226.88 mg or just under 227 mg.

b) Determine the sum of a geometric series whose terms are the masses of the antibiotic in the body after 12 tablets.
The series is:
$200 + 200(0.12) + 200(0.12)^2 + 200(0.12)^3 + \ldots + 200(0.12)^{11}$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 12, t_1 = 200, r = 0.12$

$$S_{12} = \frac{200(1 - 0.12^{12})}{1 - 0.12}$$

$$S_{12} = 227.2727\ldots$$

The mass of antibiotic in the body after 12 tablets is approximately 227.27 mg, or just over 227 mg.

THINK FURTHER

In *Example 4*, compare the masses of antibiotic remaining in the body for parts a and b. What do you notice about the masses?

For each extra tablet, the increase in mass is less than the preceding increase. The masses seem to be approaching a constant value that is slightly greater than 227 mg.

Animation

Check Your Understanding

4. A person takes tablets to cure a chest infection. Each tablet contains 500 mg of an antibiotic. About 15% of the mass of the antibiotic remains in the body when the next tablet is taken. Determine the mass of antibiotic in the body after each number of tablets:

a) 3 tablets **b)** 10 tablets

a)

Number of tablets	Mass of antibiotic (mg)
1	500
2	$500 + 500(0.15) = 575$
3	$500 + 500(0.15) + 500(0.15)^2 = 586.25$

The sum is 586.25. So, after taking the 3rd tablet, the total mass is 586.25 mg, or about 586 mg.

b) Determine the sum of this geometric series:
$500 + 500(0.15) + 500(0.15)^2 + 500(0.15)^3 + \ldots + 500(0.15)^9$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 10$, $t_1 = 500$, and $r = 0.15$

$$S_{10} = \frac{500(1 - 0.15^{10})}{1 - 0.15}$$

$$S_{10} = 588.2352\ldots$$

The mass of antibiotic is about 588 mg.

Check Your Understanding

Answers:

4. a) 586.25 mg, or just over 586 mg

b) approximately 588.24 mg, or just over 588 mg

Discuss the Ideas

1. Why do the terms in some geometric series alternate between positive and negative numbers, but the terms in an arithmetic series never alternate?

In a geometric series, when *r* is negative, the terms alternate in sign because powers of *r* with an even exponent are positive, and powers of *r* with an odd exponent are negative. In an arithmetic series, when *d* is negative, the terms decrease; when *d* is positive, the terms increase; and when *d* is 0 the terms are constant. So, the terms never alternate in sign.

2. How can you identify when a problem may be modelled by an arithmetic series or modelled by a geometric series?

If a problem involves a number and a constant that is repeatedly added or subtracted, the problem may be modelled by an arithmetic series. If the problem involves a number and a constant that is repeatedly multiplied or divided, then the problem may be modelled by a geometric series.

Exercises

A

3. Write a geometric series for each geometric sequence.

a) 1, 4, 16, 64, 256, . . .

1 + 4 + 16 + 64 + 256 + . . .

b) 20, −10, 5, −2.5, 1.25, . . .

20 − 10 + 5 − 2.5 + 1.25 − . . .

4. Which series appear to be geometric? If the series could be geometric, determine S_5.

a) $2 + 4 + 8 + 16 + 32 + \ldots$

The series could be geometric.
S_5 is: $2 + 4 + 8 + 16 + 32 = 62$

b) $2 - 4 + 8 - 16 + 32 - \ldots$

The series could be geometric.
S_5 is: $2 - 4 + 8 - 16 + 32 = 22$

c) $1 + 4 + 9 + 16 + 25 + \ldots$

The series is not geometric.

d) $-3 + 9 - 27 + 81 - 243 + \ldots$

The series could be geometric.
S_5 is: $-3 + 9 - 27 + 81 - 243 = -183$

5. Use the given data about each geometric series to determine the indicated value. Give the answers to 3 decimal places where necessary.

a) $t_1 = 1, r = 0.3$; determine S_8

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 8, t_1 = 1, r = 0.3$

$S_8 = \dfrac{1(1 - 0.3^8)}{1 - 0.3}$

$S_8 \doteq 1.428$

b) $t_1 = \frac{3}{4}, r = \frac{1}{2}$; determine S_4

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 4, t_1 = \frac{3}{4}, r = \frac{1}{2}$

$$S_4 = \frac{\frac{3}{4}\left(1 - \left(\frac{1}{2}\right)^4\right)}{1 - \frac{1}{2}}$$

$S_4 = \frac{45}{32}$, or approximately 1.406

B

6. Determine S_6 for each geometric series.

a) $2 + 10 + 50 + \ldots$

$t_1 = 2$ and r is: $\frac{10}{2} = 5$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 6, t_1 = 2, r = 5$

$S_6 = \dfrac{2(1 - 5^6)}{1 - 5}$

$S_6 = 7812$

b) $80 - 40 + 20 - \ldots$

$t_1 = 80$ and r is: $\frac{-40}{80} = -0.5$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 6, t_1 = 80, r = -0.5$

$S_6 = \dfrac{80(1 - (-0.5)^6)}{1 - (-0.5)}$

$S_6 = 52.5$

7. Determine S_{10} for each geometric series. Give the answers to 3 decimal places.

a) $0.1 + 0.01 + 0.001 + 0.0001 + \ldots$

$t_1 = 0.1$ and r is: $\frac{0.01}{0.1} = 0.1$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 10, t_1 = 0.1, r = 0.1$

$S_{10} = \dfrac{0.1(1 - 0.1^{10})}{1 - 0.1}$

$S_{10} \doteq 0.111$

b) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

$t_1 = 1$ and r is: $\dfrac{-\frac{1}{3}}{1} = -\frac{1}{3}$

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute:

$n = 10, t_1 = 1, r = -\frac{1}{3}$

$$S_{10} = \frac{1\left(1 - \left(-\frac{1}{3}\right)^{10}\right)}{1 - \left(-\frac{1}{3}\right)}$$

$S_{10} \doteq 0.750$

8. a) Explain why this series appears to be geometric:

$1 + 5 + 25 + 125 + \ldots$

After the 1st term, each term is 5 times as great as the preceding term.

TEACHER NOTE

Achievement Indicator

Questions 5, 6, and 7 address AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

TEACHER NOTE

Achievement Indicator

Question 8 addresses AI 10.1: Identify assumptions made when identifying a geometric series.

b) What information do you need to be certain that this is a geometric series?

I need to know that the series has a common ratio of 5.

c) What assumptions do you make when you identify or extend a geometric series?

I assume that the ratio of consecutive terms is the common ratio.

9. For each geometric series, determine how many terms it has then calculate its sum.

a) $1 - 2 + 4 - 8 + \ldots - 512$

$t_1 = 1$ and r is $\frac{-2}{1} = -2$

To determine n, use: $t_n = t_1 r^{n-1}$

Substitute: $t_n = -512$, $t_1 = 1$, $r = -2$

$-512 = 1(-2)^{n-1}$

$(-2)^9 = (-2)^{n-1}$

$9 = n - 1$

$n = 10$

There are 10 terms.

To determine the sum, use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 10$, $t_1 = 1$, $r = -2$

$$S_{10} = \frac{1(1 - (-2)^{10})}{1 - (-2)}$$

$S_{10} = -341$

The sum is -341.

b) $-6561 + 2187 - 729 + 243 - \ldots - 1$

$t_1 = -6561$ and r is $\frac{2187}{-6561} = -\frac{1}{3}$

To determine n, use: $t_n = t_1 r^{n-1}$

Substitute: $t_n = -1$, $t_1 = -6561$, $r = -\frac{1}{3}$

$$-1 = -6561\left(-\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{6561} = \left(-\frac{1}{3}\right)^{n-1}$$

$$\left(-\frac{1}{3}\right)^8 = \left(-\frac{1}{3}\right)^{n-1}$$

$8 = n - 1$

$n = 9$

There are 9 terms.

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 9$, $t_1 = -6561$, $r = -\frac{1}{3}$

$$S_9 = \frac{-6561\left(1 - \left(-\frac{1}{3}\right)^9\right)}{1 - \left(-\frac{1}{3}\right)}$$

$S_9 = -4921$

The sum is -4921.

10. Identify the terms in each partial sum of a geometric series.

a) $S_5 = 62, r = 2$

To determine t_1,

use $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 5, S_n = 62, r = 2$

$$62 = \frac{t_1(1 - 2^5)}{1 - 2}$$

$$62 = 31t_1$$

$$t_1 = 2$$

So, t_2 is $2(2) = 4$; t_3 is $4(2) = 8$;

t_4 is $8(2) = 16$; t_5 is $16(2) = 32$

b) $S_8 = 1111.1111; r = 0.1$

To determine t_1,

use $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 8$,

$S_n = 1111.1111, r = 0.1$

$$1111.1111 = \frac{t_1(1 - 0.1^8)}{1 - 0.1}$$

$$1111.1111 = 1.111\ 111\ 1t_1$$

$$t_1 = 1000$$

So, t_2 is $1000(0.1) = 100$;

t_3 is $100(0.1) = 10$;

t_4 is $10(0.1) = 1$;

t_5 is $1(0.1) = 0.1$;

t_6 is $(0.1)(0.1) = 0.01$;

t_7 is $0.01(0.1) = 0.001$;

t_8 is $0.001(0.1) = 0.0001$

11. On Monday, Ian had 3 friends visit his personal profile on a social networking website. On Tuesday, each of these 3 friends had 3 different friends visit Ian's profile. On Wednesday, each of the 9 friends on Tuesday had 3 different friends visit Ian's profile.

a) Write the total number of friends who visited Ian's profile as a geometric series. What is the first term? What is the common ratio?

The 1st term is 3, the number of friends on Monday.

The common ratio is 3.

So, the geometric series is: $3 + 9 + 27$

b) Suppose this pattern continued for 1 week. What is the total number of people who visited Ian's profile? How do you know your answer is correct?

The geometric series continues and has 7 terms;

one for each day of the week.

The series is: $3 + 9 + 27 + 81 + 243 + 729 + 2187$

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 7, t_1 = 3, r = 3$

$$S_7 = \frac{3(1 - 3^7)}{1 - 3}$$

$$S_7 = 3279$$

I checked my answer by using a calculator to add the seven terms.

TEACHER NOTE

Achievement Indicator

Question 11 addresses

AI 10.9:

Solve a problem that involves a geometric series.

12. Each stroke of a vacuum pump extracts 5% of the air in a 50-m^3 tank. How much air is removed after 50 strokes? Give the answer to the nearest tenth of a cubic metre.

Number of strokes	Volume removed	Volume remaining
1	$50(0.05) = 2.5$	$50(0.95) = 47.5$
2	$47.5(0.05) = 2.375$, or $2.5(0.95)$	$47.5(0.95) = 45.125$
3	$45.125(0.05) = 2.256\ 25$, or $2.5(0.95)^2$	

The volumes removed form a geometric series with 1st term 2.5 and common ratio 0.95.

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 50$, $t_1 = 2.5$, $r = 0.95$

$$S_{50} = \frac{2.5(1 - 0.95^{50})}{1 - 0.95}$$

$S_{50} = 46.1527\ldots$

After 50 strokes, about 46.2 m^3 of air is removed.

13. The sum of the first 10 terms of a geometric series is 1705. The common ratio is −2. Determine S_{11}. Explain your reasoning.

To determine t_1, use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $S_n = 1705$, $n = 10$, $r = -2$

$$1705 = \frac{t_1(1 - (-2)^{10})}{1 - (-2)}$$

$1705 = -341t_1$

$t_1 = -5$

Then, $S_{11} = S_{10} + t_{11}$

$S_{11} = 1705 + (-5)(-2)^{10}$

$S_{11} = -3415$

C

14. Binary notation is used to represent numbers on a computer. For example, the number 1111 in base two represents $1(2)^3 + 1(2)^2 + 1(2)^1 + 1$, or 15 in base ten.

a) Why is the sum above an example of a geometric series?

Each term is one-half of the preceding term.

b) Which number in base ten is represented by
11 111 111 111 111 111 111 in base two? Explain your reasoning.

There are twenty 1s digits in the number, so it can be written as the geometric series:
$1(2)^{19} + 1(2)^{18} + 1(2)^{17} + \ldots + 1(2)^{1} + 1$
This series has 20 terms, with 1st term 2^{19} and common ratio 0.5.

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 20, t_1 = 2^{19}, r = 0.5$

$S_{20} = \dfrac{2^{19}(1 - 0.5^{20})}{1 - 0.5}$

$S_{20} = 1\,048\,575$

The number is 1 048 575.

15. Show how you can use geometric series to determine this sum:
$1 + 2 + 3 + 4 + 8 + 9 + 16 + 27 + 32 + 64 + 81 + 128 + 243 + 256 + 512$

This sum comprises two geometric series:
$1 + 3 + 9 + 27 + 81 + 243$ **and**
$2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512$

For the first series

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 6, t_1 = 1, r = 3$

$S_6 = \dfrac{1(1 - 3^6)}{1 - 3}$

$S_6 = 364$

For the second series

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 9, t_1 = 2, r = 2$

$S_9 = \dfrac{2(1 - 2^9)}{1 - 2}$

$S_9 = 1022$

The sum is: $364 + 1022 = 1386$

16. Determine the common ratio of a geometric series that has these partial sums: $S_3 = -\dfrac{49}{8}, S_4 = -\dfrac{105}{16}, S_5 = -\dfrac{217}{32}$

$S_4 = S_3 + t_4$

Substitute for S_4 and S_3.

$-\dfrac{105}{16} = -\dfrac{49}{8} + t_4$

$t_4 = -\dfrac{7}{16}$

$S_5 = S_4 + t_5$

Substitute for S_5 and S_4.

$-\dfrac{217}{32} = -\dfrac{105}{16} + t_5$

$t_5 = -\dfrac{7}{32}$

$t_5 = t_4(r)$

$-\dfrac{7}{32} = -\dfrac{7}{16}(r)$

$r = \dfrac{1}{2}$

The common ratio is $\frac{1}{2}$.

Multiple-Choice Questions

1. For which geometric series is -1023 the sum to 10 terms?

A. $1 - 2 + 4 - 8 + \ldots$ **B.** $1 + 2 + 4 + 8 + \ldots$

C. $-1 + 2 - 4 + 8 - \ldots$ **D.** $-1 - 2 - 4 - 8 - \ldots$

2. The sum of the first n terms of a geometric series is: $S_n = 4^n - 1$
For this series:
I. The common ratio is 4.
II. The first 3 terms are 3, 12, and 48.
III. $S_{2n} = 2^{4n} - 1$

A. Statements I, II, and III are correct.
B. Statements I and II are correct.
C. Statements II and III are correct.
D. Statements I and III are correct.

TEACHER NOTE

Solution strategy: Substitute $n = 1$ and $n = 2$, then subtract the sums to determine t_2. Divide t_2 by t_1 to determine r. Replace n with $2n$, then simplify to get S_{2n}.

Study Note

The rule for the sum of the first n terms of a geometric series has the restriction $r \neq 1$. Identify the geometric series with first term a and $r = 1$, then determine the sum of n terms.

When the first term is a and $r = 1$, the series is $a + a + a + a + \ldots$
The sum of n terms is an.

ANSWERS

3. a) $1 + 4 + 16 + 64 + 256 + \ldots$ **b)** $20 - 10 + 5 - 2.5 + 1.25 - \ldots$
4. a) 62 **b)** 22 **d)** -183 **5. a)** approximately 1.428 **b)** approximately 1.406
6. a) 7812 **b)** 52.5 **7. a)** approximately 0.111 **b)** approximately 0.750
9. a) 10 terms; -341 **b)** 9 terms; -4921 **10. a)** 2, 4, 8, 16, 32
b) 1000, 100, 10, 1, 0.1, 0.01, 0.001, 0.0001 **11. a)** $3 + 9 + 27$; 3; 3 **b)** 3279 people
12. 46.2 m^3 **13.** -3415 **14. b)** 1 048 575 **15.** 1386 **16.** $\frac{1}{2}$

Multiple Choice
1. D **2.** A

CHECKPOINT 2

Self-Assess

Can you . . .	To check, try question . . .	For review, see . . .
write a geometric sequence and explain how you know it is geometric?		Page 32 in Lesson 1.3 (Example 2)
identify the assumptions you make when you identify a geometric sequence or series?		Page 35 in Lesson 1.3 (Discuss the Ideas)
use a rule to determine the *n*th term in a geometric sequence?	1, 2	Page 31 in Lesson 1.3 (Example 1)
use a rule to determine any of n, t_1, t_n, or r in a geometric sequence?	3	Page 33 in Lesson 1.3 (Example 3)
model and solve problems involving geometric sequences?		Page 34 in Lesson 1.3 (Example 4)
use a rule to determine the sum S_n of a geometric series?	5	Page 45 in Lesson 1.4 (Example 1)
use a rule to determine t_1 in a geometric series, given the values of r, n, or S_n?		Pages 45–46 in Lesson 1.4 (Example 2)
use a rule to determine n and S_n in a geometric series, given the values of t_1 and r?	5	Page 46 in Lesson 1.4 (Example 3)
model and solve problems involving geometric series?	6	Page 47 in Lesson 1.4 (Example 4)

Assess Your Understanding

1.3

1. Multiple Choice For this geometric sequence: −5000, 500, −50, . . .; which number below is the value of t_9?

A. 0.0005 **B.** −0.0005 **C.** 0.000 05 **D.** −0.000 05

2. This sequence is geometric: 2, −6, 18, −54, . . .

a) Write a rule to determine the *n*th term.

Use: $t_n = t_1 r^{n-1}$ **Substitute:** $t_1 = 2$, $r = -3$
$t_n = 2(-3)^{n-1}$

b) Use your rule to determine the 10th term.

Use: $t_n = 2(-3)^{n-1}$ **Substitute:** $n = 10$
$t_{10} = 2(-3)^9$
$t_{10} = -39\ 366$
The 10th term is −39 366.

TEACHER NOTE

Have students complete the Frayer diagram on Master 1.4a to summarize their knowledge of geometric sequences, then draw and complete a similar diagram for geometric series.

TEACHER NOTE

Achievement Indicator

Question 2 addresses AI 10.3: Derive a rule for determining the general term of a geometric sequence.

3. Use the given data about each geometric sequence to determine the indicated value.

a) $t_4 = -5$ and $t_7 = 135$; determine t_1

Use $t_n = t_1 r^{n-1}$ to determine r.
First substitute: $n = 4$, $t_n = -5$
$-5 = t_1 r^{4-1}$
$-5 = t_1 r^3$ ①
Then substitute: $n = 7$, $t_n = 135$
$135 = t_1 r^{7-1}$
$135 = t_1 r^6$ ②
Write equation ② as: $135 = t_1 r^3(r^3)$
From equation ①, substitute $t_1 r^3 = -5$
$135 = -5r^3$ Divide by -5.
$-27 = r^3$
$r = \sqrt[3]{-27}$
$r = -3$
Substitute $r = -3$ in equation ①.
$-5 = t_1(-3)^3$
$t_1 = \frac{5}{27}$

b) $t_1 = -1$ and $t_4 = -19\,683$; determine r

Use: $t_n = t_1 r^{n-1}$
Substitute: $n = 4$, $t_4 = -19\,683$, $t_1 = -1$
$-19\,683 = -1r^{4-1}$
$-19\,683 = -1r^3$
$19\,683 = r^3$
$\sqrt[3]{19\,683} = r$
$r = 27$

TEACHER NOTE

Achievement Indicators

Question 3 addresses AI 10.4: Determine t_1, r, n, or t_n in a problem that involves a geometric sequence.

Question 4 addresses AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

1.4

4. Multiple Choice The sum of the first 5 terms of a geometric series is $\frac{121}{81}$. The common ratio is $\frac{1}{3}$. What is the 2nd term?

A. $\frac{1}{3}$ **B.** 1 **C.** $\frac{121}{3}$ **D.** $\frac{1}{9}$

5. Use the given data about each geometric series to determine the indicated value.

a) $t_1 = -4, r = 3$; determine S_5

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 5, t_1 = -4, r = 3$

$S_5 = \frac{-4(1 - 3^5)}{1 - 3}$

$S_5 = -484$

b) $3125 + 625 + 125 + \ldots + \frac{1}{25}$; determine n

r is: $\frac{625}{3125} = \frac{1}{5}$

Use: $t_n = t_1 r^{n-1}$

Substitute: $t_n = \frac{1}{25}, t_1 = 3125, r = \frac{1}{5}$

$\frac{1}{25} = 3125\left(\frac{1}{5}\right)^{n-1}$

$\frac{1}{78\,125} = \left(\frac{1}{5}\right)^{n-1}$

$\left(\frac{1}{5}\right)^7 = \left(\frac{1}{5}\right)^{n-1}$

$7 = n - 1$

$n = 8$

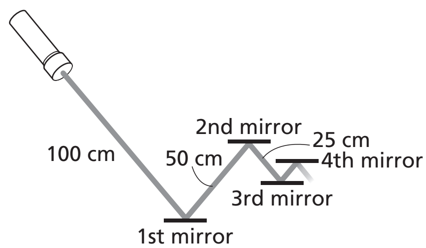
TEACHER NOTE

Achievement Indicators

Question 5 addresses AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

Question 6 addresses AI 10.9: Solve a problem that involves a geometric sequence or series.

6. The diagram shows a path of light reflected by mirrors. After the first path, the length of each path is one-half the preceding length.



a) What is the length of the path from the 4th mirror to the 5th mirror?

The lengths of the paths, in centimetres, form this geometric sequence: 100, 50, 25, 12.5, 6.25, . . .
The length of the path from the 4th mirror to the 5th mirror is the 5th term: 6.25 cm

b) To the nearest hundredth of a centimetre, what is the total length of the path from the 1st mirror to the 10th mirror?

The total length of the path is the sum of the first 10 terms of this geometric series: 100 + 50 + 25 + 12.5 + 6.25 + . . .

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ Substitute: $n = 10, t_1 = 100, r = 0.5$

$S_{10} = \frac{100(1 - 0.5^{10})}{1 - 0.5}$

$S_{10} = 199.8046\ldots$

The path is approximately 199.80 cm long.

ANSWERS

1. D 2. a) $t_n = 2(-3)^{n-1}$ b) −39 366 3. a) $\frac{5}{27}$ b) 27 4. A
5. a) −484 b) 8 6. a) 6.25 cm b) 199.80 cm

MATH LAB

1.5 Graphing Geometric Sequences and Series

Lesson Organizer
60 – 75 min

Key Math Concepts
The value of the common ratio affects the graphs of a geometric sequence and a geometric series.

Curriculum Focus

SO	AI
RF10	10.2, 10.8

Processes: PS, R, T

Student Materials
- graphing calculator, or computer with graphing software
- Master 1.2 (*optional*)
- grid paper (*optional*)

Dynamic Activity

FOCUS **Investigate the graphs of geometric sequences and geometric series.**

Get Started

Here are 4 geometric sequences:

A. $1, 2, 4, 8, 16, \ldots$ **B.** $1, -2, 4, -8, 16, \ldots$

C. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **D.** $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

Compare the sequences. How are they alike? How are they different?

All 4 sequences have 1st term 1. For Sequences A and B, the odd numbered terms are equal and the even numbered terms are opposites; so their common ratios are opposites. The same is true for Sequences C and D. The numbers in Sequence A are equal to the denominators in Sequence C, so their common ratios are reciprocals. The same is true for Sequences B and D.

Construct Understanding

Use a graphing calculator or graphing software to investigate graphs of geometric sequences and geometric series that have the same first term but different common ratios.

A. Choose a positive first term. Choose a common ratio, r, in each of the intervals in the table below. For each common ratio, create the first 5 terms of a geometric sequence.

Sample response:

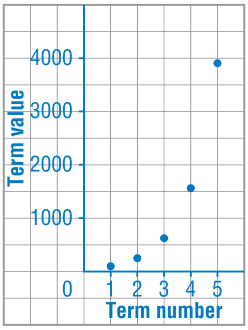
Interval	Common ratio, r	Geometric sequence
$r > 1$	$r = 2.5$	100, 250, 625, 1562.5, 3906.25, . . .
$0 < r < 1$	$r = \frac{1}{4}$	100, 25, 6.25, 1.5625, 0.390 625, . . .
$-1 < r < 0$	$r = -0.2$	100, −20, 4, −0.8, 0.16, . . .
$r < -1$	$r = -1.5$	100, −150, 225, −337.5, 506.25, . . .

B. For each sequence

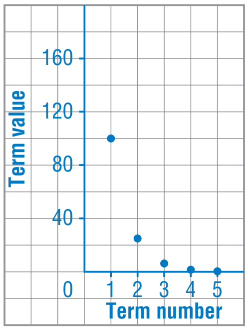
- Graph the term numbers on the horizontal axis and the term values on the vertical axis. Sketch and label each graph on a grid below, or print each graph.

Sample response:

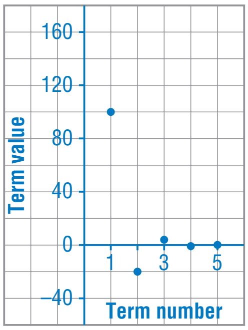
Geometric Sequence, $r = 2.5$



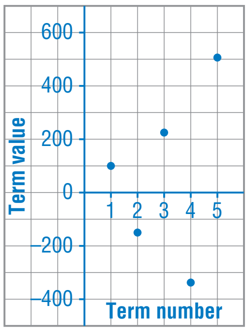
Geometric Sequence, $r = \frac{1}{4}$



Geometric Sequence, $r = -0.2$



Geometric Sequence, $r = -1.5$



- What happens to the term values as more points are plotted?

As more points are plotted:

For $r > 1$, the term values increase.

For $0 < r < 1$, the term values decrease and approach 0.

For $-1 < r < 0$, the term values alternate between positive and negative, and approach 0.

For $r < -1$, the term values alternate between positive and negative, and increase in numerical value.

TEACHER NOTE

Encourage students to use the terms *convergent* and *divergent* to describe the geometric sequences.

TEACHER NOTE

Achievement Indicator

Construct Understanding, Parts A to D, prepares students for a more formal treatment, in Lesson 1.6, related to AI 10.8: Explain why a geometric series is convergent or divergent.

TEACHER NOTE

DI: Common Difficulties

Some students may not see how the value of the common ratio affects the graphs of geometric sequences and series. Suggest a methodical approach: start with the same first term and create two pairs of examples, using common ratios of 2 and –2, 0.5 and –0.5.

TEACHER NOTE

DI: Extending Thinking

Have students investigate the "rate of convergence" as r approaches 0. For example, students could create tables of values and graphs for sequences and series where $r = 0.9$, $r = 0.8$, $r = 0.7$, and so on. They should deduce that the lesser the value of r, the "faster" the convergence.

C. Use the four geometric sequences in Part A to create four corresponding geometric series.

For each series

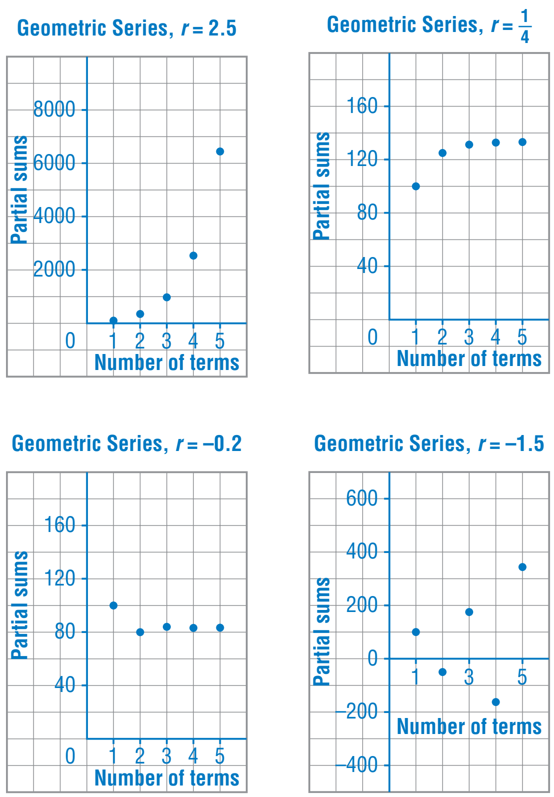
- Complete the table below by calculating these partial sums: S_1, S_2, S_3, S_4, S_5

Sample response:

Interval	Common ratio, r	S_1	S_2	S_3	S_4	S_5
$r > 1$	$r = 2.5$	100	350	975	2537.5	6443.75
$0 < r < 1$	$r = \frac{1}{4}$	100	125	131.25	132.8125	133.203 125
$-1 < r < 0$	$r = -0.2$	100	80	84	83.2	83.36
$r < -1$	$r = -1.5$	100	−50	175	−162.5	343.75

- Graph the numbers of terms in the partial sums on the horizontal axis and the partial sums on the vertical axis. Sketch and label each graph on a grid below, or print each graph.

Sample response:



- What happens to the partial sums as more points are plotted?

As more points are plotted:

For $r > 1$, the partial sums increase.

For $0 < r < 1$, the partial sums increase and appear to approach a constant value close to 133.

For $-1 < r < 0$, the partial sums increase then decrease, and appear to approach a constant value close to 83.

For $r < -1$, the partial sums alternate between positive and negative values; the positive terms increase and the negative terms decrease.

D. Without graphing

- Describe the graph of this geometric sequence: $3, 2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots$

The common ratio is $\frac{2}{3}$, which is between 0 and 1, so the term values decrease and approach 0.

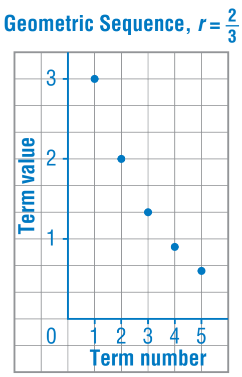
- Describe the graph of the partial sums of this geometric series:

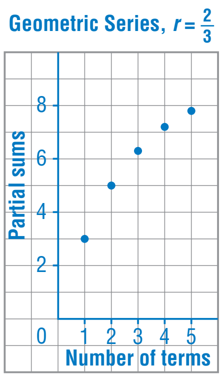
$3 + 2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \ldots$

The common ratio is $\frac{2}{3}$, which is between 0 and 1, so the partial sums increase and approach a constant value.

Verify your descriptions by graphing. Sketch and label each graph on a grid below, or print each graph.

Sample response:





Assess Your Understanding

TEACHER NOTE

Achievement Indicator

Question 1 addresses AI 10.2: Provide and justify an example of a geometric sequence.

1. Create the first 5 terms of a geometric sequence with positive first term for each description of a graph.

a) The term values approach 0 as more points are plotted.

For the term values to approach 0, the common ratio must be between 0 and 1; for example, with $r = 0.4$, a possible sequence is: 2, 0.8, 0.32, 0.128, 0.0512, . . .

b) The term values increase as more points are plotted.

For the term values to increase, the common ratio must be greater than 1; for example, with $r = 4$, a possible sequence is: 2, 8, 32, 128, 512, . . .

c) The term values alternate between positive and negative as more points are plotted.

For the term values to alternate between positive and negative, the common ratio must be negative; for example, with $r = -0.4$, a possible sequence is: 2, −0.8, 0.32, −0.128, 0.0512, . . .

2. Create a geometric series with positive first term for each description of a graph.

a) The partial sums approach a constant value as more points are plotted.

For the partial sums to approach a constant value, the common ratio must be between −1 and 1; for example, with $r = 0.4$, a possible series is: $2 + 0.8 + 0.32 + 0.128 + 0.0512 + \dots$

b) The partial sums increase as more points are plotted.

For the partial sums to increase, the common ratio must be greater than 1; for example, with $r = 4$, a possible series is: $2 + 8 + 32 + 128 + 512 + \dots$

1.6 Infinite Geometric Series

FOCUS **Determine the sum of an infinite geometric series.**

Get Started

Write $0.\overline{6}$ as a series.
What type of series is it? How do you know?

Sample response: $0.\overline{6} = 0.6666...$
$= 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$

This is a geometric series because, after the 1st term, each term is $\frac{1}{10}$ of the preceding term.

Construct Understanding

Draw a square.
Divide it into 4 equal squares, then shade 1 smaller square.
Divide one smaller square into 4 equal squares, then shade 1 square.
Continue dividing smaller squares into 4 equal squares and shading 1 square, for as long as you can.
Suppose you could continue this process indefinitely.
Estimate the total area of the shaded squares. Explain your reasoning.

I created a sequence for the areas of the shaded squares.
I wrote each partial sum, S_n, of the related series as a decimal.

Square, n	Area of square	S_n
1	$\frac{1}{4}$	0.25
2	$\frac{1}{16}$	0.3125
3	$\frac{1}{64}$	0.328 125
4	$\frac{1}{256}$	0.332 031 25
5	$\frac{1}{1024}$	0.333 007 8125

If I could continue the process indefinitely, the decimal that would represent the total area of the shaded squares would be 0.3333 . . ., or $0.\overline{3}$.
So, I estimate that the total area of the shaded squares is $\frac{1}{3}$.

Lesson Organizer
75 min

Key Math Concepts
An infinite geometric series that converges has a finite sum.

Curriculum Focus

SO	AI
RF10	10.7, 10.8, 10.9

Processes: PS, R, T

Student Materials
- scientific calculator
- graphing calculator (*optional*)

Vocabulary
infinite geometric series, sum to infinity

TEACHER NOTE

DI: Extending Thinking
Have students determine the ratio of the area of the shaded squares to the area of the unshaded squares, then use the ratio to determine the sum of the areas of each type of square. *(For each shaded square there are 2 unshaded squares with the same area; this indicates a 2 : 1 ratio. Taking the area of the original square as 1 unit, the sum of the areas of the shaded squares is $\frac{1}{3}$.)*

An **infinite geometric series** has an infinite number of terms.
For an infinite geometric series, if the sequence of partial sums converges to a constant value as the number of terms increases, then the geometric series is convergent and the constant value is the finite sum of the series. This sum is called the **sum to infinity** and is denoted by S_∞.

Example 1 Estimating the Sum of an Infinite Geometric Series

Predict whether each infinite geometric series has a finite sum.
Estimate each finite sum.

a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

b) $0.5 + 1 + 2 + 4 + \ldots$

c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

SOLUTION

For each geometric series, calculate some partial sums.

a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$S_1 = \frac{1}{2}$

$S_2 = S_1 + \frac{1}{4}$
$S_2 = \frac{1}{2} + \frac{1}{4}$
$S_2 = \frac{3}{4}$

$S_3 = S_2 + \frac{1}{8}$
$S_3 = \frac{3}{4} + \frac{1}{8}$
$S_3 = \frac{7}{8}$

$S_4 = S_3 + \frac{1}{16}$
$S_4 = \frac{7}{8} + \frac{1}{16}$
$S_4 = \frac{15}{16}$

The next term in the series is $\frac{1}{32}$.

$S_5 = S_4 + \frac{1}{32}$
$S_5 = \frac{15}{16} + \frac{1}{32}$
$S_5 = \frac{31}{32}$

These partial sums appear to get closer to 1.
An estimate of the finite sum is 1.

b) $0.5 + 1 + 2 + 4 + \ldots$

$S_1 = 0.5$

$S_2 = S_1 + 1$
$S_2 = 0.5 + 1$
$S_2 = 1.5$

$S_3 = S_2 + 2$
$S_3 = 1.5 + 2$
$S_3 = 3.5$

$S_4 = S_3 + 4$
$S_4 = 3.5 + 4$
$S_4 = 7.5$

As the number of terms increases, the partial sums increase, so the series does not have a finite sum.

Check Your Understanding

1. Predict whether each infinite geometric series has a finite sum. Estimate each finite sum.

a) $\frac{1}{3} + \frac{1}{12} + \frac{1}{48} + \frac{1}{192} + \ldots$

b) $-4 - 8 - 16 - 32 - \ldots$

c) $\frac{1}{10} - \frac{1}{100} + \frac{1}{1000} - \frac{1}{10\,000} + \ldots$

a) The next term in the series is $\frac{1}{768}$.

$S_1 = \frac{1}{3}$, or $0.\bar{3}$

$S_2 = S_1 + \frac{1}{12}$, or $0.41\bar{6}$

$S_3 = S_2 + \frac{1}{48}$, or 0.4375

$S_4 = S_3 + \frac{1}{192}$, or approximately 0.4427

$S_5 = S_4 + \frac{1}{768}$, or approximately 0.4440

An estimate of the finite sum is $0.\bar{4}$.

b) The next term in the series is −64.

$S_1 = -4$

$S_2 = S_1 - 8$, or -12

$S_3 = S_2 - 16$, or -28

$S_4 = S_3 - 32$, or -60

$S_5 = S_4 - 64$, or -124

The partial sums decrease, so the series does not have a finite sum.

Check Your Understanding

Answers:

1. a) $0.\bar{4}$

b) does not have a finite sum

c) $0.\overline{09}$

c) $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

$S_1 = \frac{1}{2}$

$S_1 = 0.5$

$S_2 = S_1 - \frac{1}{4}$

$S_2 = \frac{1}{2} - \frac{1}{4}$

$S_2 = \frac{1}{4}$

$S_2 = 0.25$

$S_3 = S_2 + \frac{1}{8}$

$S_3 = \frac{1}{4} + \frac{1}{8}$

$S_3 = \frac{3}{8}$

$S_3 = 0.375$

$S_4 = S_3 - \frac{1}{16}$

$S_4 = \frac{3}{8} - \frac{1}{16}$

$S_4 = \frac{5}{16}$

$S_4 = 0.3125$

The next term is $\frac{1}{32}$.

$S_5 = S_4 + \frac{1}{32}$

$S_5 = \frac{5}{16} + \frac{1}{32}$

$S_5 = \frac{11}{32}$

$S_5 = 0.343\ 75$

The next term is $-\frac{1}{64}$.

$S_6 = S_5 - \frac{1}{64}$

$S_6 = \frac{11}{32} - \frac{1}{64}$

$S_6 = \frac{21}{64}$

$S_6 = 0.328\ 125$

The partial sums alternately increase and decrease, but appear to get close to 0.33.
An estimate of the finite sum is 0.33.

c) The next term in the series is $\frac{1}{100\ 000}$.

$S_1 = \frac{1}{10}$, or 0.1

$S_2 = S_1 - \frac{1}{100}$, or 0.09

$S_3 = S_2 + \frac{1}{1000}$, or 0.091

$S_4 = S_3 - \frac{1}{10\ 000}$, or 0.0909

$S_5 = S_4 + \frac{1}{100\ 000}$, or 0.09091

An estimate of the finite sum is $0.\overline{09}$.

THINK FURTHER

In *Example 1*, what other strategy could you use to estimate each finite sum?

I could graph the partial sums to see which constant value they approach.

In *Example 1*, why is each partial sum written as a decimal?

It is easier to identify the number to which the series appears to converge when the partial sums are written as decimals.

In *Example 1*, each series has the same first term but different common ratios. So, from *Example 1* and Lesson 1.5, it appears that the value of r determines whether an infinite geometric series converges or diverges.

Extra Material

Graphical representations for $r = \frac{1}{10}$ and $r = -\frac{1}{8}$

Consider the rule for the sum of n terms.
For a geometric series,

$$S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$$

When $-1 < r < 1$, r^n approaches 0 as n increases indefinitely.

So, S_n approaches $\frac{t_1(1 - 0)}{1 - r}$

and, $S_\infty = \frac{t_1}{1 - r}$

THINK FURTHER

Why does this rule not apply when $r \leq -1$ or $r \geq 1$?

When $r \leq -1$ or $r \geq 1$, the infinite geometric series diverges and does not have a finite sum.

The Sum of an Infinite Geometric Series

For an infinite geometric series with first term, t_1, and common ratio, $-1 < r < 1$, the sum of the series, S_∞, is:

$$S_\infty = \frac{t_1}{1 - r}$$

Check Your Understanding

2. Determine whether each infinite geometric series converges or diverges. If it converges, determine its sum.

a) $32 + 8 + 2 + 0.5 + \ldots$

b) $100 - 10 + 1 - 0.1 + \ldots$

a) $t_1 = 32$ and r is: $\frac{8}{32} = \frac{1}{4}$

Since $-1 < r < 1$, the series converges.

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{32}{1 - \frac{1}{4}}$$

$$S_\infty = 42.\overline{6}$$

The sum is $42.\overline{6}$.

b) $t_1 = 100$ and r is:

$$\frac{-10}{100} = -0.1$$

Since $-1 < r < 1$, the series converges.

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{100}{1 - (-0.1)}$$

$$S_\infty = 90.\overline{90}$$

The sum is $90.\overline{90}$.

Example 2 Determining the Sum of an Infinite Geometric Series

Determine whether each infinite geometric series converges or diverges. If it converges, determine its sum.

a) $27 - 9 + 3 - 1 + \ldots$ **b)** $4 - 8 + 16 - 32 + \ldots$

SOLUTION

a) $27 - 9 + 3 - 1 + \ldots$

$t_1 = 27$ and r is: $\frac{-9}{27} = -\frac{1}{3}$

The common ratio is between -1 and 1, so the series converges.

Use the rule for the sum of an infinite geometric series:

$$S_\infty = \frac{t_1}{1 - r} \qquad \text{Substitute: } t_1 = 27,\ r = -\frac{1}{3}$$

$$S_\infty = \frac{27}{1 - \left(-\frac{1}{3}\right)}$$

$$S_\infty = \frac{27}{1 + \frac{1}{3}}$$

$$S_\infty = \frac{27}{\frac{4}{3}}$$

$$S_\infty = 27\left(\frac{3}{4}\right)$$

$$S_\infty = 20.25$$

The sum of the infinite geometric series is 20.25.

b) $4 - 8 + 16 - 32 + \ldots$

$t_1 = 4$ and r is: $\frac{-8}{4} = -2$

The common ratio is not between -1 and 1, so the series diverges.

The infinite geometric series does not have a finite sum.

THINK FURTHER

In *Example 2*, how could you check that the sum in part a is reasonable?

I could determine a partial sum for a large number of terms such as S_{20}. This sum should be very close to the value for S_∞.

Check Your Understanding

Answers:

2. a) converges; the sum is $42.\overline{6}$

b) converges; the sum is $90.\overline{90}$

Example 3	Using an Infinite Geometric Series to Solve a Problem

Determine a fraction that is equal to $0.4\overline{9}$.

SOLUTION

The repeating decimal $0.4\overline{9}$ can be expressed as:

$0.4 + 0.09 + 0.009 + 0.0009 + \ldots$

The repeating digits form an infinite geometric series.
The series converges because $-1 < r < 1$. Use the rule for S_∞.

$$S_\infty = \frac{t_1}{1-r} \qquad \text{Substitute: } t_1 = 0.09, \text{ or } \frac{9}{100}; r = 0.1, \text{ or } \frac{1}{10}$$

$$S_\infty = \frac{\frac{9}{100}}{1-\frac{1}{10}}$$

$$S_\infty = \frac{\frac{9}{100}}{\frac{9}{10}}, \text{ or } \frac{1}{10}$$

Add $\frac{1}{10}$ to 0.4, or $\frac{4}{10}$, the non-repeating part of the decimal:

$$\frac{4}{10} + \frac{1}{10} = \frac{5}{10}, \text{ or } \frac{1}{2}$$

So, $0.4\overline{9} = \frac{1}{2}$

Check Your Understanding

3. Determine a fraction that is equal to $0.1\overline{6}$.

$0.1\overline{6} = 0.1 + 0.06 + 0.006 + 0.0006 + \ldots$

The repeating digits form an infinite geometric series with $t_1 = 0.06$, or $\frac{6}{100}$ and $r = 0.1$, or $\frac{1}{10}$.

Use $S_\infty = \frac{t_1}{1-r}$

Substitute for t_1 and r.

$$S_\infty = \frac{\frac{6}{100}}{1-\frac{1}{10}}$$

$$S_\infty = \frac{\frac{6}{100}}{\frac{9}{10}}, \text{ or } \frac{2}{30}$$

Add: $\frac{1}{10} + \frac{2}{30} = \frac{5}{30}$, or $\frac{1}{6}$

So, $0.1\overline{6} = \frac{1}{6}$

Discuss the Ideas

1. How do you determine whether an infinite geometric series diverges or converges?

I determine the common ratio, r. If r is between -1 and 1, the series converges; if $r \leq -1$ or $r \geq 1$, the series diverges.

TEACHER NOTE

Achievement Indicator

Question 1 addresses AI 10.8: Explain why a geometric series is convergent or divergent.

2. An infinite geometric series has first term 5. Why does the series diverge when $r = 1$ or $r = -1$?

When $r = 1$, all the terms of the series are equal and the partial sum will increase if the first term is positive and decrease if the first term is negative. When $r = -1$, the terms of the series have the same numerical value, but alternate in sign; so the partial sums alternate between a value equal to the first term and 0.

Check Your Understanding

Answer:

3. $\frac{1}{6}$

Exercises

A

3. Determine whether each infinite geometric series has a finite sum. How do you know?

a) $2 + 3 + 4.5 + 6.75 + \ldots$

r is: $\frac{3}{2} = 1.5$, so the sum is not finite.

b) $-0.5 - 0.05 - 0.005 - 0.0005 - \ldots$

r is: $\frac{-0.05}{-0.5} = 0.1$, so the sum is finite.

c) $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \frac{27}{128} + \ldots$

r is: $\dfrac{-\frac{3}{8}}{\frac{1}{2}} = -\frac{3}{4}$, so the sum is finite.

d) $0.1 + 0.2 + 0.4 + 0.8 + \ldots$

r is: $\frac{0.2}{0.1} = 2$, so the sum is not finite.

4. Write the first 4 terms of each infinite geometric series.

a) $t_1 = -4, r = 0.3$

$t_1 = -4$
t_2 is: $-4(0.3) = -1.2$
t_3 is: $-1.2(0.3) = -0.36$
t_4 is: $-0.36(0.3) = -0.108$

b) $t_1 = 1, r = -0.25$

$t_1 = 1$
t_2 is: $1(-0.25) = -0.25$
t_3 is: $-0.25(-0.25) = 0.0625$
t_4 is: $0.0625(-0.25) = -0.015\,625$

c) $t_1 = 4, r = \frac{1}{5}$

$t_1 = 4$
t_2 is: $4\left(\frac{1}{5}\right) = \frac{4}{5}$
t_3 is: $\frac{4}{5}\left(\frac{1}{5}\right) = \frac{4}{25}$
t_4 is: $\frac{4}{25}\left(\frac{1}{5}\right) = \frac{4}{125}$

d) $t_1 = -\frac{3}{2}, r = -\frac{3}{8}$

$t_1 = -\frac{3}{2}$
t_2 is: $-\frac{3}{2}\left(-\frac{3}{8}\right) = \frac{9}{16}$
t_3 is: $\frac{9}{16}\left(-\frac{3}{8}\right) = -\frac{27}{128}$
t_4 is: $-\frac{27}{128}\left(-\frac{3}{8}\right) = \frac{81}{1024}$

5. Each infinite geometric series converges. Determine each sum.

a) $8 + 2 + 0.5 + 0.125 + \ldots$

Use: $S_\infty = \dfrac{t_1}{1 - r}$

Substitute: $t_1 = 8$, $r = \dfrac{1}{4}$

$S_\infty = \dfrac{8}{1 - \frac{1}{4}}$

$S_\infty = 10.\overline{6}$

The sum is $10.\overline{6}$.

b) $-1 - \dfrac{3}{4} - \dfrac{9}{16} - \dfrac{27}{64} - \ldots$

Use: $S_\infty = \dfrac{t_1}{1 - r}$

Substitute: $t_1 = -1$, $r = \dfrac{3}{4}$

$S_\infty = \dfrac{-1}{1 - \frac{3}{4}}$

$S_\infty = -4$

The sum is -4.

c) $10 - \dfrac{20}{3} + \dfrac{40}{9} - \dfrac{80}{27} + \ldots$

Use: $S_\infty = \dfrac{t_1}{1 - r}$

Substitute: $t_1 = 10$, $r = -\dfrac{2}{3}$

$S_\infty = \dfrac{10}{1 - \left(-\frac{2}{3}\right)}$

$S_\infty = 6$

The sum is 6.

d) $-2 + \dfrac{2}{3} - \dfrac{2}{9} + \dfrac{2}{27} - \ldots$

Use: $S_\infty = \dfrac{t_1}{1 - r}$

Substitute: $t_1 = -2$, $r = -\dfrac{1}{3}$

$S_\infty = \dfrac{-2}{1 - \left(-\frac{1}{3}\right)}$

$S_\infty = -1.5$

The sum is -1.5.

TEACHER NOTE

Achievement Indicator

Question 5 addresses AI 10.6: Determine S_n in a problem that involves a geometric series.

B

6. What do you know about the common ratio of an infinite geometric series whose sum is finite?

The common ratio is less than 1 and greater than -1.

TEACHER NOTE

DI: Extending Thinking

Have students consider geometric series with first term 0. What series are possible? How does this affect your answer to question 6? *(The only possible series is 0, 0, 0, ... The sum of the series is 0. So, a geometric series with finite sum and first term 0 can have a common ratio that is any real number.)*

7. Use the given data about each infinite geometric series to determine the indicated value.

a) $t_1 = 21$, $S_\infty = 63$; determine r

Substitute for t_1 and S_∞ in $S_\infty = \dfrac{t_1}{1 - r}$.

$63 = \dfrac{21}{1 - r}$

$63 - 63r = 21$

$63r = 42$

$r = \dfrac{42}{63}$, or $\dfrac{2}{3}$

b) $r = -\dfrac{3}{4}$, $S_\infty = \dfrac{24}{7}$; determine t_1

Substitute for r and S_∞ in $S_\infty = \dfrac{t_1}{1 - r}$.

$\dfrac{24}{7} = \dfrac{t_1}{1 - \left(-\frac{3}{4}\right)}$

$\dfrac{24}{7}\left(\dfrac{7}{4}\right) = t_1$

$t_1 = 6$

8. Use an infinite geometric series to express each repeating decimal as a fraction.

a) $0.4\overline{97}$

$0.4\overline{97} = 0.4 + 0.097 + 0.000\,97 + \ldots$

This is an infinite geometric series with

$t_1 = \frac{97}{1000}$ and

r is $\frac{0.000\,97}{0.097} = \frac{1}{100}$

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{\frac{97}{1000}}{1 - \frac{1}{100}}$$

$$S_\infty = \frac{\frac{97}{1000}}{\frac{99}{100}}, \text{ or } \frac{97}{990}$$

Add: $\frac{4}{10} + \frac{97}{990} = \frac{493}{990}$

So, $0.4\overline{97} = \frac{493}{990}$

b) $1.\overline{143}$

$1.\overline{143} = 1 + 0.143 + 0.000\,143 + \ldots$

This is an infinite geometric series with

$t_1 = \frac{143}{1000}$ and

r is $\frac{0.000\,143}{0.143} = \frac{1}{1000}$

Substitute for t_1 and r in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{\frac{143}{1000}}{1 - \frac{1}{1000}}$$

$$S_\infty = \frac{\frac{143}{1000}}{\frac{999}{1000}}, \text{ or } \frac{143}{999}$$

Add: $1 + \frac{143}{999} = \frac{1142}{999}$

So, $1.\overline{143} = \frac{1142}{999}$

TEACHER NOTE

DI: Common Difficulties

For students having difficulty expressing a repeating decimal as an infinite geometric series, suggest they write 3 or more repetitions of the repeating period, then record the decimal expansion in a place-value chart. From there, they can use place value (powers of 10) to identify the terms in the infinite geometric series.

9. The hour hand on a clock is pointing to 12. The hand is rotated clockwise 180°, then another 60°, then another 20°, and so on. This pattern continues.

a) Which number would the hour hand approach if this rotation continued indefinitely? Explain what you did.

The angles, in degrees, that the hand rotates through form a geometric sequence with $t_1 = 180$ and $r = \frac{1}{3}$. The total angle turned through is the related infinite geometric series:

$$180 + \frac{180}{3} + \frac{180}{3^2} + \frac{180}{3^3} + \ldots$$

Use: $S_\infty = \frac{t_1}{1 - r}$ Substitute: $t_1 = 180$, $r = \frac{1}{3}$

$$S_\infty = \frac{180}{1 - \frac{1}{3}}$$

$$S_\infty = 270$$

When the hour hand has rotated 270° clockwise from 12, it will point to 9. So, if the rotation continued indefinitely, the hour hand would approach 9.

b) What assumptions did you make?

I assumed that the angle measures formed an infinite geometric series that converged.

TEACHER NOTE

Achievement Indicator

Questions 9, 10, and 11 address AI 10.9:
Solve a problem that involves a geometric series.

10. Brad has a balance of \$500 in a bank account. Each month he spends 40% of the balance remaining in the account.

a) Express the total amount Brad spends in the first 4 months as a series. Is the series geometric? Explain.

After:	Amount spent	Amount remaining
1 month	$\$500(0.4) = \$200$	$\$500(0.6) = \$300$
2 months	$\$300(0.4) = \$120$, or $\$500(0.6)(0.4)$	$\$300(0.6) = \$180$, or $\$500(0.6)^2$
3 months	$\$180(0.4) = \$72$, or $\$500(0.6)^2(0.4)$	$\$180(0.6) = \$108$, or $\$500(0.6)^3$
4 months	$\$108(0.4) = \$43.20$, or $\$500(0.6)^3(0.4)$	

The amounts spent are:
$\$500(0.4) + \$500(0.6)(0.4) + \$500(0.6)^2(0.4) + \$500(0.6)^3(0.4)$
This is a geometric series with $t_1 = \$500(0.4)$ and $r = 0.6$

b) Determine the approximate amount Brad spends in 10 months. Explain what you did.

The amount Brad spends in 10 months is the sum of the first 10 terms of the series in part a.

Use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}, r \neq 1$

Substitute: $n = 10$, $t_1 = 200$, $r = 0.6$

$S_{10} = \dfrac{200(1 - 0.6^{10})}{1 - 0.6}$

$S_{10} = 496.9766...$

Brad spends about \$496.98 in 10 months.

c) Suppose Brad could continue this pattern of spending indefinitely. Would he eventually empty his bank account? Explain.

No, because Brad can only spend money in dollars and cents, and not fractions of a cent, so each amount he spends will be rounded to the nearest cent. Continuing the pattern of spending 40% each month, and rounding to the nearest cent each time, Brad will eventually end up with \$0.01 remaining in his account. Since 40% of \$0.01 is less than 1 penny, this amount will never be spent.

C

11. Write the product of $0.\overline{a}$ and $0.\overline{b}$ as a fraction, where a and b represent 1-digit natural numbers. Explain your strategy.

$0.\overline{a} = 0.a + 0.0a + 0.00a + \ldots$

This is an infinite geometric series with $t_1 = 0.a$, or $\frac{a}{10}$ and $r = 0.1$, or $\frac{1}{10}$.

Use: $S_\infty = \frac{t_1}{1 - r}$ Substitute: $t_1 = \frac{a}{10}$, $r = \frac{1}{10}$

$$S_\infty = \frac{\frac{a}{10}}{1 - \left(\frac{1}{10}\right)}$$

$$S_\infty = \frac{\frac{a}{10}}{\frac{9}{10}}$$

$$S_\infty = \frac{a}{9}$$

So, $0.\overline{a} = \frac{a}{9}$; similarly, $0.\overline{b} = \frac{b}{9}$; then $(0.\overline{a})(0.\overline{b})$ is

$$\left(\frac{a}{9}\right)\left(\frac{b}{9}\right) = \frac{ab}{81}$$

12. Create 2 different infinite geometric series with a sum of 4. Explain what you did.

Sample response: Choose a value for r between -1 and 1, then determine a value for t_1.

Let $r = -0.25$.

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute:

$S_\infty = 4$, $r = -0.25$

$$4 = \frac{t_1}{1 - (-0.25)}$$

$t_1 = 5$

Let $r = 0.6$.

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute:

$S_\infty = 4$, $r = 0.6$

$$4 = \frac{t_1}{1 - 0.6}$$

$t_1 = 1.6$

One series is: $5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \ldots$

Another series is: $1.6 + 0.96 + 0.576 + 0.3456 + \ldots$

13. Determine the sum of this infinite geometric series:

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{32}} + \frac{1}{\sqrt{128}} + \frac{1}{\sqrt{512}} + \ldots$$

The common ratio is r: $\dfrac{\frac{1}{\sqrt{8}}}{\frac{1}{\sqrt{2}}} = \dfrac{1}{\sqrt{4}}$, or $\dfrac{1}{2}$

Use: $S_\infty = \dfrac{t_1}{1 - r}$ **Substitute: $t_1 = \dfrac{1}{\sqrt{2}}$, $r = \dfrac{1}{2}$**

$$S_\infty = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}$$

$$S_\infty = \frac{2}{\sqrt{2}}$$

The sum is $\dfrac{2}{\sqrt{2}}$.

Multiple-Choice Questions

1. What is the sum of this infinite geometric series?

$$10 - \frac{20}{3} + \frac{40}{9} - \frac{80}{27} + \ldots$$

A. 30 **B.** 4 **C.** 6 **D.** 20

2. Which infinite geometric series has the sum $-8.\overline{3}$?

A. $-5 - 2 - 0.8 - 0.32 - \ldots$ **B.** $-5 + 2 - 0.8 + 0.32 - \ldots$

C. $5 - 2 + 0.8 - 0.32 + \ldots$ **D.** $5 + 2 + 0.8 + 0.32 + \ldots$

3. How many of these geometric series have finite sums?

$1 + 0.5 + 0.125 + 0.0625 + \ldots$ $3 - 9 + 27 - 81 + \ldots$

$1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \ldots$ $-12 - 6 - 3 - 1.5 - \ldots$

A. 1 series **B.** 2 series **C.** 3 series **D.** 4 series

TEACHER NOTE

Solution strategy: The answer cannot be parts C or D because an inspection of these series shows that their sums will be positive. So, determine S_∞ for the series in parts A and B.

Study Note

What is a rule for determining the sum of an infinite geometric series? When is it appropriate to apply this rule? When is it not appropriate?

The rule for the sum of an infinite geometric series, S_∞, is $S_\infty = \dfrac{t_1}{1 - r}$, where t_1 is the first term and r is the common ratio. This rule may be applied when r is between -1 and 1. The rule may not be applied when $r \le -1$ or $r \ge 1$.

ANSWERS

3. a) not finite **b)** finite **c)** finite **d)** not finite **4. a)** $-4 - 1.2 - 0.36 - 0.108$ **b)** $1 - 0.25 + 0.0625 - 0.015\,625$ **c)** $4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125}$ **d)** $-\frac{3}{2} + \frac{9}{16} - \frac{27}{128} + \frac{81}{1024}$ **5. a)** $10.\overline{6}$ **b)** -4 **c)** 6 **d)** -1.5 **7. a)** $\frac{2}{3}$ **b)** 6 **8. a)** $\frac{493}{990}$ **b)** $\frac{1142}{999}$ **9. a)** 9 **10. a)** $\$500(0.4) + \$500(0.6)(0.4) + \$500(0.6)^2(0.4) + \$500(0.6)^3(0.4)$; geometric **b)** \$496.98 **c)** yes **11.** $\frac{ab}{81}$ **13.** $\frac{2}{\sqrt{2}}$

Multiple Choice

1. C **2.** A **3.** B

STUDY GUIDE

Concept Summary

Big Ideas	Applying the Big Ideas
• An arithmetic sequence is related to a linear function and is created by repeatedly adding a constant to an initial number. An arithmetic series is the sum of the terms of an arithmetic sequence.	This means that: • The common difference of an arithmetic sequence is equal to the slope of the line through the points of the graph of the related linear function. • Rules can be derived to determine the *n*th term of an arithmetic sequence and the sum of the first *n* terms of an arithmetic series.
• A geometric sequence is created by repeatedly multiplying an initial number by a constant. A geometric series is the sum of the terms of a geometric sequence.	• The common ratio of a geometric sequence can be determined by dividing any term after the first term by the preceding term. • Rules can be derived to determine the *n*th term of a geometric sequence and the sum of the first *n* terms of a geometric series.
• Any finite series has a sum, but an infinite geometric series may or may not have a sum.	• The common ratio determines whether an infinite series has a finite sum.

Chapter Study Notes

- What information do you need to know about an arithmetic sequence and a geometric sequence to determine t_n?

For an arithmetic sequence, I need to know the first term, t_1, and the common difference, d. Then I can use the rule: $t_n = t_1 + d(n - 1)$

For a geometric sequence, I need to know the first term, t_1, and the common ratio, r. Then I can use the rule: $t_n = t_1 r^{n-1}$

- What information do you need to know about an arithmetic series and a geometric series to determine the sum S_n?

For an arithmetic series, I need to know the first term, t_1, the common difference, d, and the number of terms, n.

Then I can use the rule: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$

Alternatively, for an arithmetic series, I need to know the first term, t_1, the number of terms, n, and the last term, t_n. Then I can use the rule:

$$S_n = \frac{n(t_1 + t_n)}{2}$$

For a geometric series, I need to know the first term, t_1, the common ratio, r, and the number of terms, n. Then I can use the rule:

$$S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$$

Skills Summary

Skill	Description	Example
Determine the general term, t_n, for an arithmetic sequence. **(1.1, 1.2)** Question 1	A rule is: $t_n = t_1 + d(n - 1)$ where t_1 is the first term, d is the common difference, and n is the number of terms.	For this arithmetic sequence: $-9, -3, 3, 9, \ldots$ the 20th term is: $t_{20} = -9 + 6(20 - 1)$ $t_{20} = -9 + 6(19)$ $t_{20} = 105$
Determine the sum of n terms, S_n, for an arithmetic series. **(1.2)** Question 2	When n is the number of terms, t_1 is the first term, t_n is the nth term, and d is the common difference One rule is: $S_n = \frac{n(t_1 + t_n)}{2}$ Another rule is: $S_n = \frac{n[2t_1 + d(n - 1)]}{2}$	For this arithmetic series: $5 + 7 + 9 + 11 + 13 + 15 + 17$; the sum of the first 7 terms is: $S_7 = \frac{7(5 + 17)}{2}$ $S_7 = \frac{7(22)}{2}$ $S_7 = 77$
Determine the general term, t_n, for a geometric sequence. **(1.3, 1.4)**	A rule is: $t_n = t_1r^{n-1}$ where t_1 is the first term, r is the common ratio, and n is the number of terms.	For this geometric sequence: $1, -0.25, 0.0625, \ldots$ the 6th term is: $t_6 = (-0.25)^{6-1}$ $t_6 = (-0.25)^5$ $t_6 = -0.000\,976\,5\ldots$
Determine the sum of n terms, S_n, of a geometric series. **(1.4)** Question 5	A rule is: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ where t_1 is the first term, r is the common ratio, and n is the number of terms.	For this geometric series: $4, 2, 1, \ldots$ the sum of the first 10 terms is: $S_{10} = \frac{4(1 - 0.5^{10})}{1 - 0.5}$ $S_{10} = 7.9921\ldots$, or approximately 8
Determine the sum, S_∞, of a convergent infinite geometric series. **(1.6)** Question 7	When r is between -1 and 1, use this rule: $S_\infty = \frac{t_1}{1 - r}$ where t_1 is the first term and r is the common ratio.	For this geometric series: $100 - 50 + 25 - \ldots$ the sum is: $S_\infty = \frac{100}{1 - (-0.5)}$ $S_\infty = 66.\overline{6}$

REVIEW

1.1

1. During the 2003 fire season, the Okanagan Mountain Park fire was the most significant wildfire event in B.C. history. By September 7, the area burned had reached about 24 900 ha and the fire was spreading at a rate of about 150 ha/h.

   a) Suppose the fire continued to spread at the same rate. Create terms of a sequence to represent the area burned for each of the next 6 h. Why is the sequence arithmetic?

   Each hour, the area increases by 150 ha. So, for each of the next 6 h, the area burned in hectares is:
   25 050, 25 200, 25 350, 25 500, 25 650, 25 800, . . .
   This sequence is arithmetic because the difference between consecutive terms is constant.

   b) Write a rule for the general term of the sequence in part a. Use the rule to predict the area burned after 24 h. What assumptions did you make?

   Use: $t_n = t_1 + d(n - 1)$ **Substitute:** $t_1 = 25\,050$, $d = 150$
   The general term is: $t_n = 25\,050 + 150(n - 1)$
   Substitute: $n = 24$
   $t_{24} = 25\,050 + 150(24 - 1)$
   $t_{24} = 28\,500$
   After 24 h, the area burned was 28 500 ha; I assumed that the fire continued to spread at the same rate.

TEACHER NOTE

Achievement Indicators

Question 1 addresses AI 9.1: Identify the assumption(s) made when defining an arithmetic sequence.
AI 9.2: Provide and justify an example of an arithmetic sequence.
AI 9.3: Derive a rule for determining the general term of an arithmetic sequence.

Question 2 addresses AI 9.7: Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.

1.2

2. Use the given data about each arithmetic series to determine the indicated value.

   a) $5 + 3\frac{1}{2} + 2 + \frac{1}{2} - 1 - \ldots$; determine S_{21}

   Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$
   Substitute:
   $n = 21$, $t_1 = 5$, $d = -1.5$
   $S_{21} = \dfrac{21[2(5) - 1.5(21 - 1)]}{2}$
   $S_{21} = -210$

   b) $S_{12} = 78$ and $t_1 = -21$; determine t_{12}

   Use: $S_n = \dfrac{n(t_1 + t_n)}{2}$
   Substitute:
   $n = 12$, $S_{12} = 78$, $t_1 = -21$
   $78 = \dfrac{12(-21 + t_{12})}{2}$
   $78 = 6(-21 + t_{12})$
   $13 = -21 + t_{12}$
   $t_{12} = 34$

1.3

3. Explain the meaning of this newspaper headline.

I-Pod Sales Grew Geometrically from 2001 to 2006

In 2002, the number of sales was equal to a constant multiplied by the number of sales in 2001. In 2003, the number of sales was equal to the same constant multiplied by the number of sales in 2002. This pattern continued up to 2006.

4. A soapstone carving was appraised at $2500. The value of the carving is estimated to increase by 12% each year. What will be the approximate value of the carving after 15 years?

The values of the carving, in dollars, form a geometric sequence with $t_1 = 2500$ and $r = 1.12$. The value, in dollars, after 15 years is t_{16}.

Use $t_n = t_1r^{n-1}$ Substitute: $n = 16$, $t_1 = 2500$, $r = 1.12$

$$t_{16} = 2500(1.12)^{15}$$

$$t_{16} = 13\,683.9144\ldots$$

After 15 years, the carving will be worth approximately $13 684.

TEACHER NOTE

Achievement Indicators

Question 3 addresses AI 10.2: Provide and justify an example of a geometric sequence.

Question 4 addresses AI 10.9: Solve a problem that involves a geometric sequence.

Question 5 addresses AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

1.4

5. Determine the sum of the geometric series below. Give the answer to 3 decimal places.

$$-700 + 350 - 175 + \ldots + 5.468\,75$$

Use: $t_n = t_1r^{n-1}$ to determine n.

Substitute: $t_n = 5.468\,75$, $t_1 = -700$, r is: $\dfrac{350}{-700} = -0.5$

$$5.468\,75 = -700(-0.5)^{n-1}$$

$$-0.007\,812\,5 = (-0.5)^{n-1}$$

$$(-0.5)^7 = (-0.5)^{n-1}$$

$$n = 8$$

Then, use: $S_n = \dfrac{t_1(1 - r^n)}{1 - r}$, $r \neq 1$

Substitute: $n = 8$, $t_1 = -700$, $r = -0.5$

$$S_8 = \frac{-700(1 - (-0.5)^8)}{1 - (-0.5)}$$

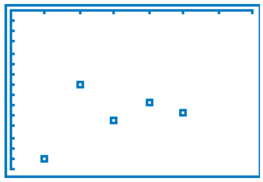
$$S_8 \doteq -464.844$$

1.5

6. Use a graphing calculator or graphing software.

Use the series from question 5. Graph the first 5 partial sums. Explain how the graph shows whether the series converges or diverges.

Sample response: The series has $t_1 = -700$ and $r = -0.5$; its partial sums are: $-700, -350, -525, -437.5, -481.25, \ldots$
The series converges because the points appear to approach a constant value of approximately -450.



TEACHER NOTE

Achievement Indicators

Questions 6 and 7 address
AI 10.8: Explain why a geometric series is convergent or divergent.

Question 8 addresses
AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.
AI 10.8: Explain why a geometric series is convergent or divergent.

1.6

7. Explain how you can use the common ratio of a geometric series to identify whether the series is convergent or divergent.

A geometric series with a common ratio less than 1 and greater than -1 converges. A geometric series with a common ratio less than or equal to -1 or greater than or equal to 1 diverges.

8. Identify each infinite geometric series that converges. Determine the sum of any series that converges.

a) $2 - 3 + 4.5 - 6.75 + \ldots$

r is $\frac{-3}{2} = -1.5$,
so the series diverges.

b) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots$

r is $\frac{\frac{2}{9}}{\frac{1}{3}} = \frac{2}{3}$,
so the series converges.

Use: $S_\infty = \frac{t_1}{1 - r}$

Substitute: $t_1 = \frac{1}{3}$, $r = \frac{2}{3}$

$S_\infty = \frac{\frac{1}{3}}{1 - \frac{2}{3}}$, or 1

9. A small steel ball bearing is moving vertically between two electromagnets whose relative strength varies each second. The ball bearing moves 10 cm up in the 1st second, then 5 cm down in the 2nd second, then 2.5 cm up in 3rd second, and so on. This pattern continues.

TEACHER NOTE

Achievement Indicator

Question 9 addresses AI 10.9: Solve a problem that involves a geometric series.

a) Assume the distance the ball bearing moves up is positive; the distance it moves down is negative.

i) Write a series to represent the distance travelled in 5 s.

Each second, the distance is halved.
In the first 5 s, the distance in centimetres is:
$10 - 5 + 2.5 - 1.25 + 0.625$

ii) Calculate the sum of the series. What does this sum represent?

Use: $S_n = \frac{t_1(1 - r^n)}{1 - r}, r \neq 1$ **Substitute:** $n = 5, t_1 = 10, r = -0.5$

$$S_5 = \frac{10(1 - (-0.5)^5)}{1 - (-0.5)}$$

$$S_5 = 6.875$$

After 5 s, the ball bearing is 6.875 cm above the magnet from which it started.

b) Suppose this process continues indefinitely. What is the sum of the series?

The series is infinite and converges.

Use: $S_\infty = \frac{t_1}{1 - r}$ **Substitute:** $t_1 = 10, r = -0.5$

$$S_\infty = \frac{10}{1 - (-0.5)}$$

$$S_\infty = 6.\overline{6}$$

The sum of the series is $6.\overline{6}$ **cm.**

ANSWERS

1. a) 25 050, 25 200, 25 350, 25 500, 25 650, 25 800, . . . **b)** approximately 28 500 ha
2. a) −210 **b)** 34 **4.** $13 684 **5.** approximately −464.844 **8. a)** diverges
b) converges; 1 **9. a) i)** $10 - 5 + 2.5 - 1.25 + 0.625$ **ii)** 6.875 cm **b)** $6.\overline{6}$ cm

PRACTICE TEST

1. Multiple Choice What is the sum of the first 30 terms of this arithmetic series? $-5 - 2 + 1 + 4 + \ldots$

A. 1152 **B.** 1155 **C.** 1158 **D.** 1161

2. Multiple Choice What is the sum of the first 10 terms of this geometric series? $-12\,800 + 6400 - 3200 + 1600 - \ldots$

A. 8525 **B.** −8525 **C.** −8537.5 **D.** 8537.5

3. a) Which sequence below appears to be arithmetic? Justify your answer.

i) 4, −10, 16, −22, 28, . . . **ii)** 4, −10, −24, −38, −52, . . .

In part i, the differences of consecutive terms are: −14, 26, −38, 50. Since these differences are not equal, the sequence is not arithmetic.
In part ii, the differences of consecutive terms are: −14, −14, −14, −14. Since these differences are equal, the sequence appears to be arithmetic.

b) Assume that the sequence you identified in part a is arithmetic. Determine:

i) a rule for t_n **ii)** t_{17}

The arithmetic sequence is:
4, −10, −24, −38, −52, . . .
Use: $t_n = t_1 + d(n - 1)$
Substitute: $t_1 = 4$, $d = -14$
$t_n = 4 - 14(n - 1)$

Use: $t_n = 4 - 14(n - 1)$
Substitute: $n = 17$
$t_{17} = 4 - 14(17 - 1)$
$t_{17} = -220$

iii) the term that has value −332

Use: $t_n = 4 - 14(n - 1)$ **Substitute:** $t_n = -332$
$-332 = 4 - 14(n - 1)$
$24 = n - 1$
$n = 25$
The 25th term has value −332.

TEACHER NOTE

Achievement Indicators

Question 1 addresses AI 9.7: Determine t_1, d, n, or S_n in a problem that involves an arithmetic series.

Question 2 addresses AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

Question 3 addresses
AI 9.1: Identify the assumption(s) made when defining an arithmetic sequence.
AI 9.3: Derive a rule for determining the general term of an arithmetic sequence.
AI 9.5: Determine t_1, d, n, or t_n in a problem that involves an arithmetic sequence.

4. For a geometric sequence, $t_4 = -1000$ and $t_7 = 1$; determine:

a) t_1

Use: $t_7 = t_4r^3$
Substitute:
$t_7 = 1, t_4 = -1000$
$1 = -1000r^3$
$r^3 = \frac{1}{-1000}$
$r = -\frac{1}{10}$, or -0.1
Use: $t_n = t_1r^{n-1}$
Substitute:
$n = 7, t_7 = 1, r = -0.1$
$1 = t_1(-0.1)^{7-1}$
$t_1 = \frac{1}{(-0.1)^6}$
$t_1 = 1\ 000\ 000$

b) the term with value 0.0001

Use: $t_n = t_1r^{n-1}$
Substitute:
$t_n = 0.0001, t_1 = 1\ 000\ 000,$
$r = -0.1$
$0.0001 = 1\ 000\ 000(-0.1)^{n-1}$
$0.000\ 000\ 000\ 1 = (-0.1)^{n-1}$
$(-0.1)^{10} = (-0.1)^{n-1}$
$n = 11$
The 11th term has value 0.0001.

5. a) For the infinite geometric series below, identify which series converges and which series diverges. Justify your answer.

i) $100 - 150 + 225 - 337.5 + \ldots$

The common ratio, r, is: $\frac{-150}{100} = -1.5$
Since r is less than -1, the series diverges.

ii) $10 + 5 + 2.5 + 1.25 + \ldots$

r is: $\frac{5}{10} = \frac{1}{2}$
Since r is between -1 and 1, the series converges.

b) For which series in part a can you determine its sum? Explain why, then determine this sum.

I can determine the sum of an infinite geometric series that converges; that is, the series in part a ii).

Use: $S_\infty = \frac{t_1}{1-r}$ Substitute: $t_1 = 10, r = \frac{1}{2}$, or 0.5
$S_\infty = \frac{10}{1-0.5}$, or 20
The sum of the series is 20.

TEACHER NOTE

Achievement Indicators

Question 4 addresses AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.

Question 5 addresses
AI 10.6: Determine t_1, r, n, or S_n in a problem that involves a geometric series.
AI 10.7: Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.
AI 10.8: Explain why a geometric series is convergent or divergent.

6. This sequence represents the approximate lengths in centimetres of a spring that is stretched by loading it with from one to four 5-kg masses: 50, 54, 58, 62, . . .
Suppose the pattern in the sequence continues. What will the length of the spring be when it is loaded with ten 5-kg masses? Explain how you found out.

Since the differences between consecutive terms are equal, then the series appears to be arithmetic. The length of the spring, in centimetres, will be the 10th term of the arithmetic sequence.

Use: $t_n = t_1 + d(n - 1)$ **Substitute:** $n = 10$, $t_1 = 50$, $d = 4$

$t_n = 50 + 4(10 - 1)$

$t_n = 86$

The spring will be 86 cm long.

TEACHER NOTE

Achievement Indicator

Questions 6 and 7 address AI 9.8: Solve a problem that involves an arithmetic sequence or series.

7. As part of his exercise routine, Earl uses a program designed to help him eventually do 100 consecutive push-ups. He started with 17 push-ups in week 1 and planned to increase the number of push-ups by 2 each week.

a) In which week does Earl expect to reach his goal?

The number of push-ups each week form an arithmetic sequence with $t_1 = 17$ **and** $d = 2$**. Determine** n **for** $t_n = 100$**.**

Use: $t_n = t_1 + d(n - 1)$ **Substitute:** $t_n = 100$, $t_1 = 17$, $d = 2$

$100 = 17 + 2(n - 1)$

$83 = 2n - 2$

$2n = 85$

$n = 42.5$

Earl should reach his goal in the 43rd week.

b) What is the total number of push-ups he will have done when he reaches his goal? Explain how you know.

The total number of push-ups is the sum of the first 43 terms of the arithmetic sequence.

Use: $S_n = \dfrac{n[2t_1 + d(n - 1)]}{2}$ **Substitute:** $n = 43$, $t_1 = 17$, $d = 2$

$S_{43} = \dfrac{43[2(17) + 2(43 - 1)]}{2}$

$S_{43} = 2537$

Earl will have done 2537 push-ups when he reaches his goal.

ANSWERS

1. B **2.** B **3. a) i)** not arithmetic **ii)** arithmetic **b) i)** $t_n = 4 - 14(n - 1)$ **ii)** -220 **iii)** t_{25} **4. a)** 1 000 000 **b)** t_{11} **5. a) i)** diverges **ii)** converges **b)** 20 **6.** 86 cm **7. a)** week 43 **b)** 2537