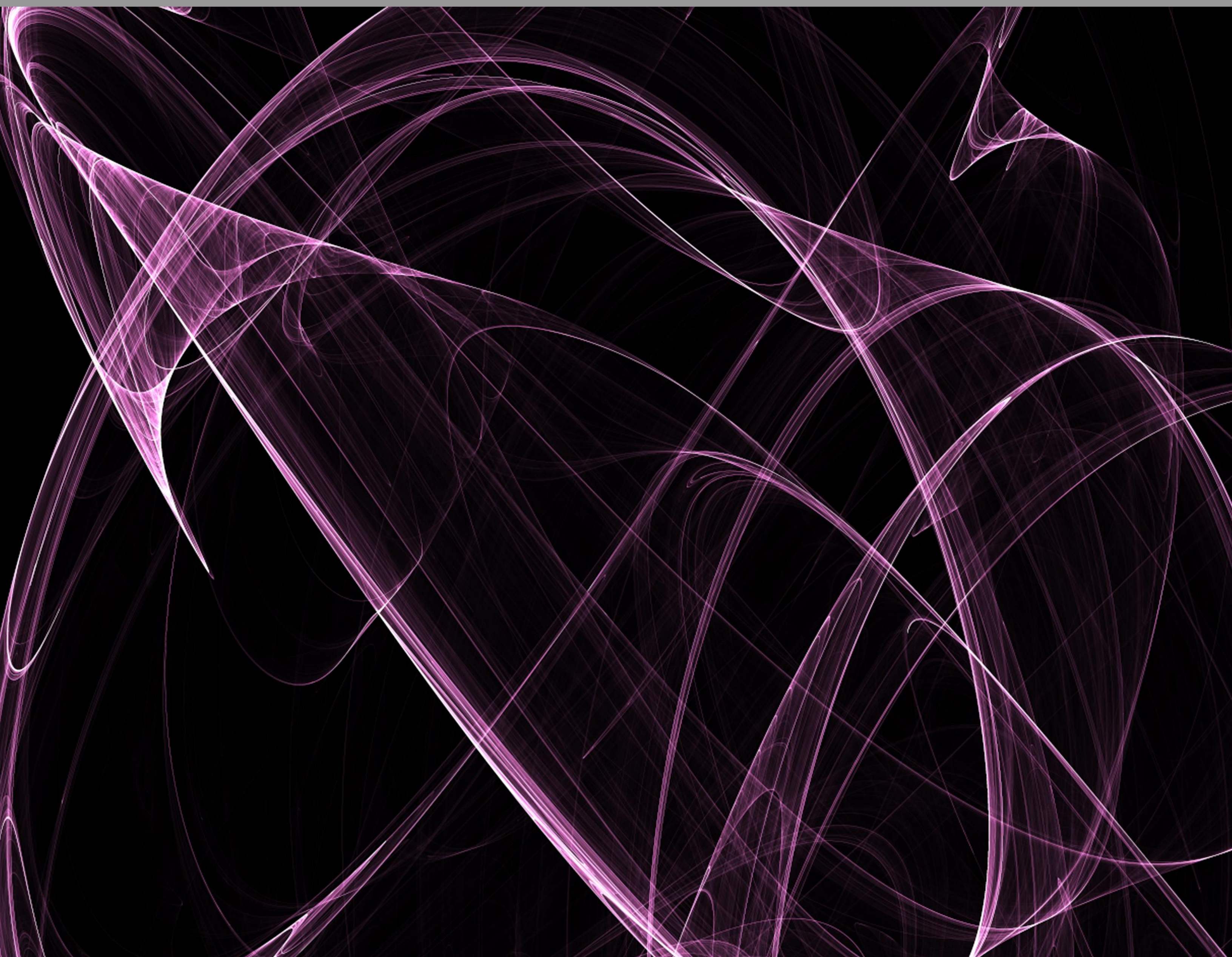
CK-12 Calculus



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Contents

CHAPTER 1 Functions, Limits, and Continuity

Chapter Outline

1.1 Equations and Graphs

Learning Objectives

A student will be able to:

- Find solutions of graphs of equations.
- Find key properties of graphs of equations including intercepts and symmetry.
- Find points of intersections of two equations.
- Interpret graphs as models.

Introduction

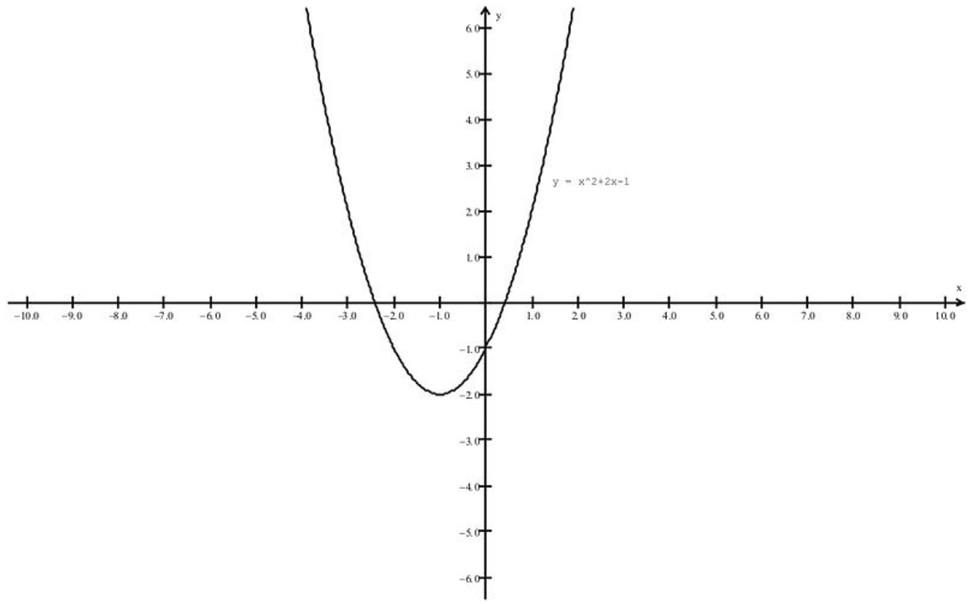
In this lesson we will review what you have learned in previous classes about mathematical equations of relationships and corresponding graphical representations and how these enable us to address a range of mathematical applications. We will review key properties of mathematical relationships that will allow us to solve a variety of problems. We will examine examples of how equations and graphs can be used to model real-life situations.

Let's begin our discussion with some examples of algebraic equations:

Example 1: $y = x^2 + 2x - 1$ The equation has ordered pairs of numbers (x, y) as solutions. Recall that a particular pair of numbers is a solution if direct substitution of the x and y values into the original equation yields a true equation statement. In this example, several solutions can be seen in the following table:

x	$y = x^2 + 2x - 1$
-4	7
-3	2
-2	-1
-1	-2
0	-1
1	2
2	7
3	14

We can graphically represent the relationships in a rectangular coordinate system, taking the x as the horizontal axis and the y as the vertical axis. Once we plot the individual solutions, we can draw the curve through the points to get a sketch of the graph of the relationship:



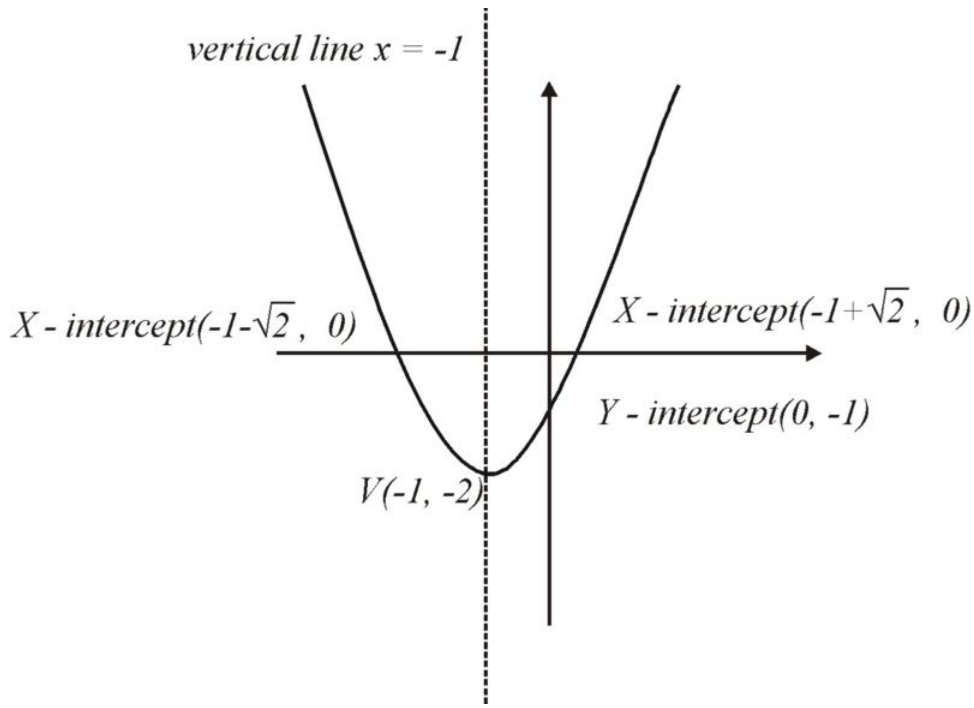
We call this shape a parabola and every quadratic function, $f(x) = ax^2 + bx + c, a \neq 0$ has a parabola-shaped graph. Let's recall how we analytically find the key points on the parabola. The vertex will be the lowest point, which for this parabola is $(-1, -2)$. In general, the vertex is located at the point $(-\frac{b}{2a}, f(-\frac{b}{2a}))$. We then can identify points crossing the x and y axes. These are called the intercepts. The $y-$intercept is found by setting $x = 0$ in the equation, and then solving for y as follows:

$y = 0^2 + 2(0) - 1 = -1$. The $y-$intercept is located at $(0, -1)$.

The $x-$intercept is found by setting $y = 0$ in the equation, and solving for x as follows: $0 = x^2 + 2x - 1$

Using the quadratic formula, we find that $x = -1 \pm \sqrt{2}$. The $x-$intercepts are located at $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$.

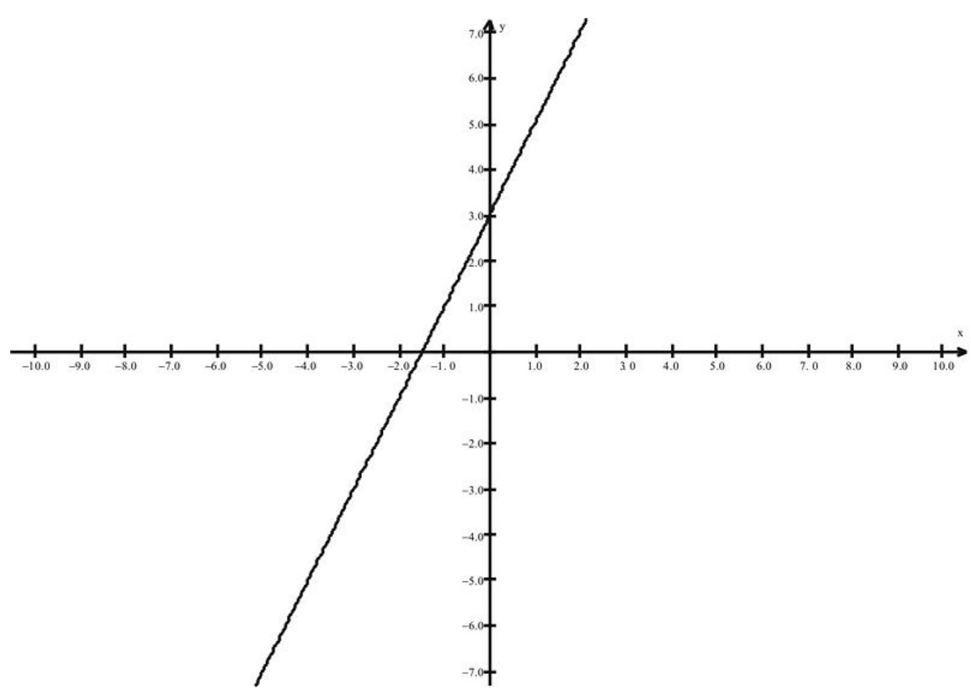
All parabolas also have a line of symmetry. This parabola has a vertical line of symmetry at $x = -1$. In general, the line of symmetry for a parabola will always pass through its vertex, and so will always be located at $x = -\frac{b}{2a}$. The graph with all of its key characteristics is summarized below:



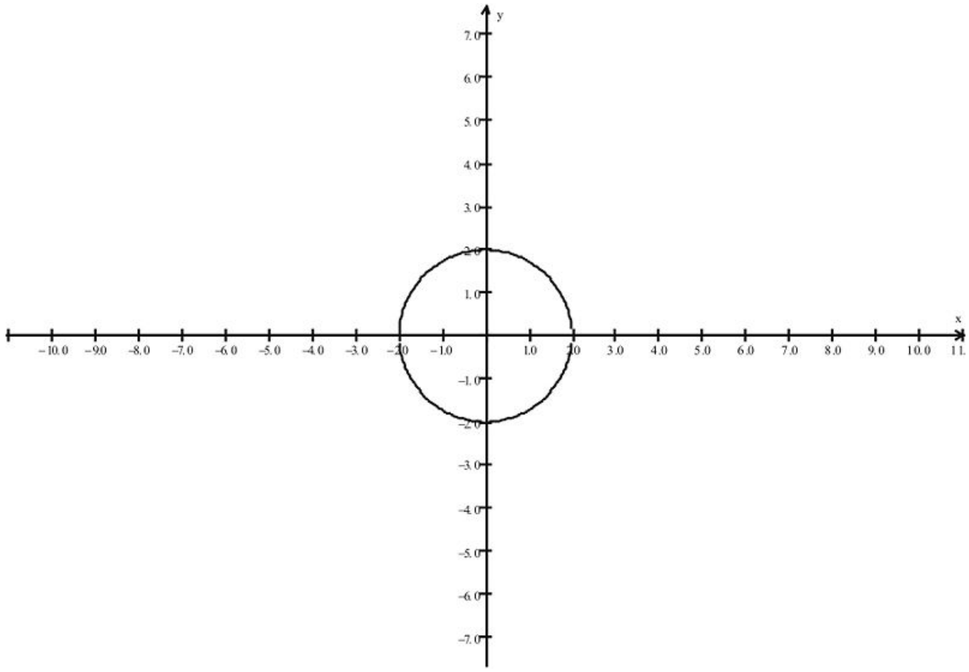
Example 2:

Here are some other examples of equations with their corresponding graphs:

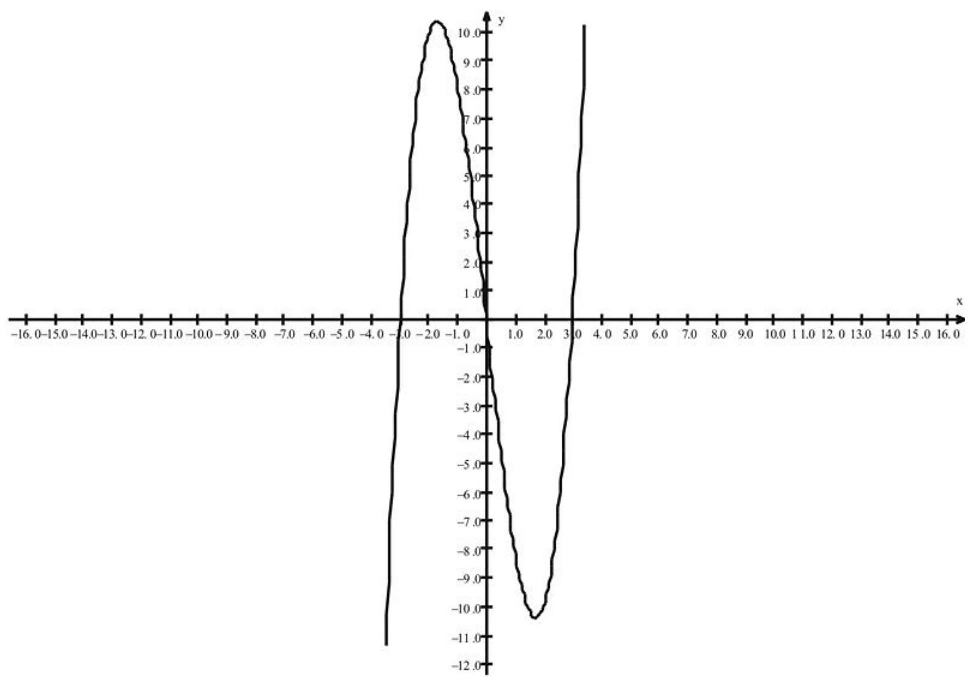
$y = 2x + 3$



$x^2 + y^2 = 4$



$y = x^3 - 9x$



Example 3:

Notice that the first equation in Example 2 is linear, so its graph is a straight line. Can you determine the intercepts?

Solution:

$x-$intercept at $(-3/2, 0)$ and $y-$intercept at $(0, 3)$.

Example 4:

Recall from pre-calculus that the second equation in Example 2 is that of a circle with center $(0,0)$ and radius $r = 2$. Can you show analytically that the radius is 2?

Solution:

Find the four intercepts, by setting $x = 0$ and solving for y, and then setting $y = 0$ and solving for x.

Example 5:

The third equation from Example 2 is an example of a polynomial relationship. Can you find the intercepts analytically?

Solution:

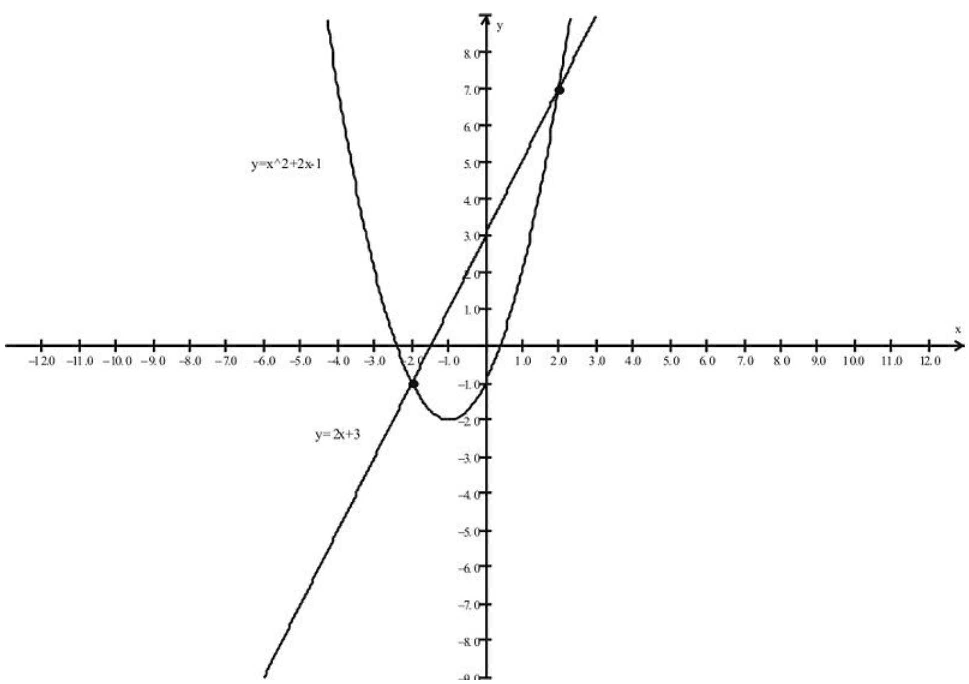
We can find the $x-$intercepts analytically by setting $y = 0$ and solving for x. So, we have

$$\begin{aligned} x^3 - 9x &= 0 \\ x(x^2 - 9) &= 0 \\ x(x-3)(x+3) &= 0 \\ x = 0, x = -3, x &= 3. \end{aligned}$$

The $x-$intercepts are located at $(-3,0),(0,0)$, and $(3,0)$. Note that $(0,0)$ is also the $y-$intercept. The $y-$intercepts can be found by setting $x = 0$. So, we have

$$\begin{aligned} x^3 - 9x &= y \\ (0)^3 - 9(0) &= y \\ y &= 0. \end{aligned}$$

Sometimes we wish to look at pairs of equations and examine where they have common solutions. Consider the linear and quadratic graphs of the previous examples. We can sketch them on the same axes:



We can see that the graphs intersect at two points. It turns out that we can solve the problem of finding the points of intersections analytically and also by using our graphing calculator. Let's review each method.

Analytical Solution

Since the points of intersection are on each graph, we can use substitution, setting the general $y-$coordinates equal to each other, and solving for x.

$$\begin{aligned}2x+3 &= x^2+2x-1\\ 0 &= x^2-4\\ x=2, x &= -2.\end{aligned}$$

We substitute each value of x into one of the original equations and find the points of intersections at $(-2,-1)$ and $(2,7)$.

Graphing Calculator Solution

Once we have entered the relationships on the **Y=** menu, we press **2nd [CALC]** and choose #5 **Intersection** from the menu. We then are prompted with a cursor by the calculator to indicate which two graphs we want to work with. Respond to the next prompt by pressing the left or right arrows to move the cursor near one of the points of intersection and press **[ENTER]**. Repeat these steps to find the location of the second point.

We can use equations and graphs to model real-life situations. Consider the following problem.

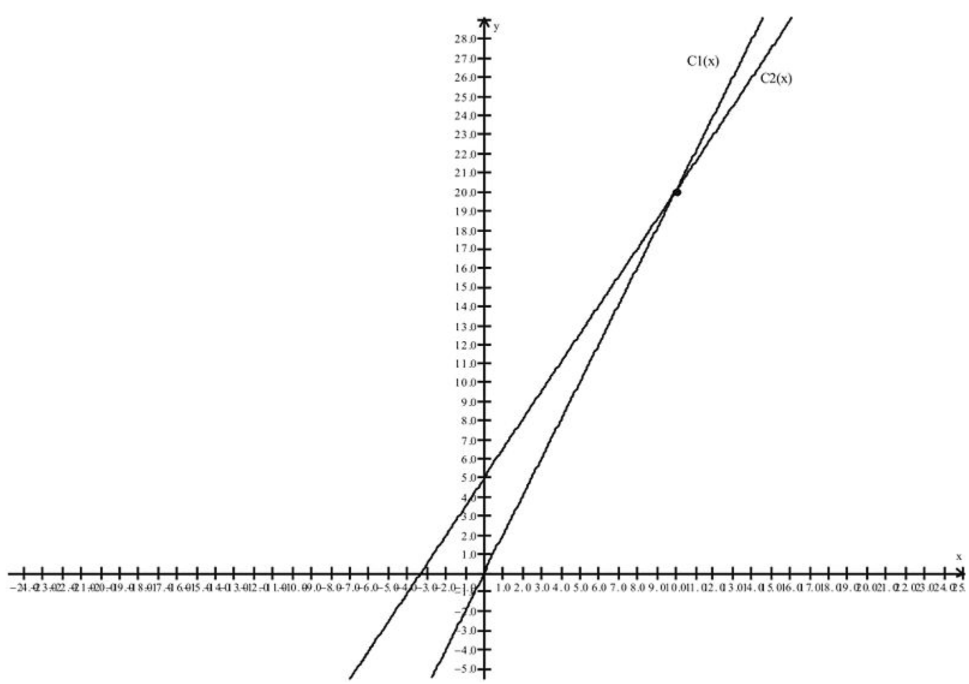
Example 6: Linear Modeling

The cost to ride the commuter train in Chicago is $2. Commuters have the option of buying a monthly coupon book costing $5 that allows them to ride the train for $1.5 on each trip. Is this a good deal for someone who commutes every day to and from work on the train?

Solution:

We can represent the cost of the two situations, using the linear equations and the graphs as follows:

$$\begin{aligned}C_1(x) &= 2x\\ C_2(x) &= 1.5x+5\end{aligned}$$



As before, we can find the point of intersection of the lines, or in this case, the break-even value in terms of trips, by solving the equation:

$$\begin{aligned} C_1(x) &= C_2(x) \\ 2x &= 1.5x+5 \\ x &= 10. \end{aligned}$$

So, even though it costs more to begin with, after 10 trips the cost of the coupon book pays off and from that point on, the cost is less than for those riders who did not purchase the coupon book.

Example 7: Non-Linear Modeling

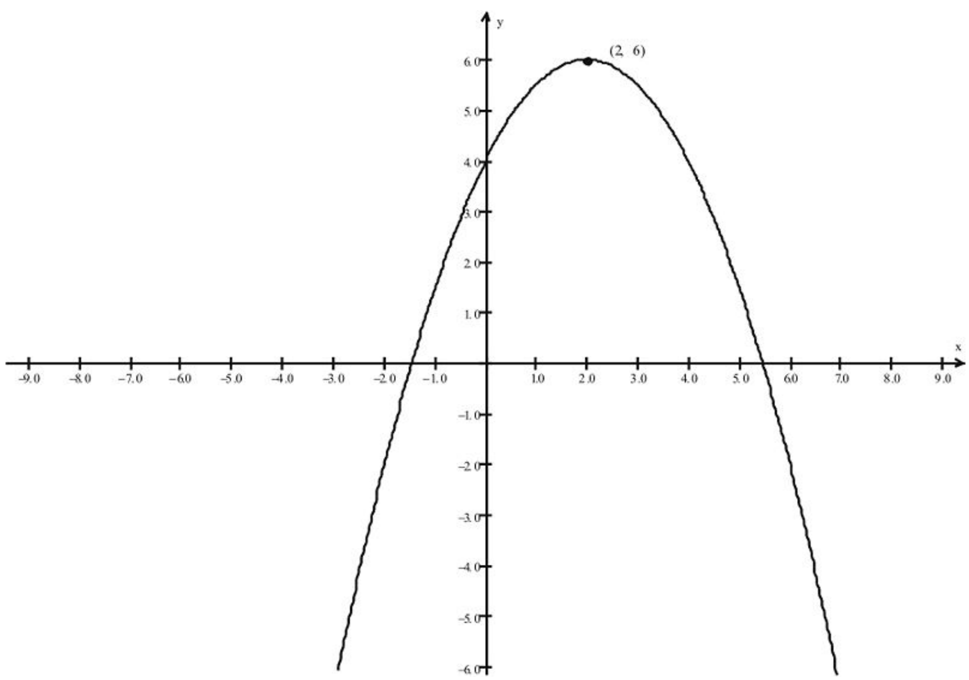
The cost of disability benefits in the Social Security program for the years 2000 - 2005 can be modeled as a quadratic function. The formula

$$Y = -0.5x^2 + 2x + 4$$

indicates the number of people Y, in millions, receiving Disability Benefits x years after 2000. In what year did the greatest number of people receive benefits? How many people received benefits in that year?

Solution:

We can represent the graph of the relationship using our graphing calculator.



The vertex is the maximum point on the graph and is located at $(2,6)$. Hence in year 2002 a total of 6 million people received benefits.

Lesson Summary

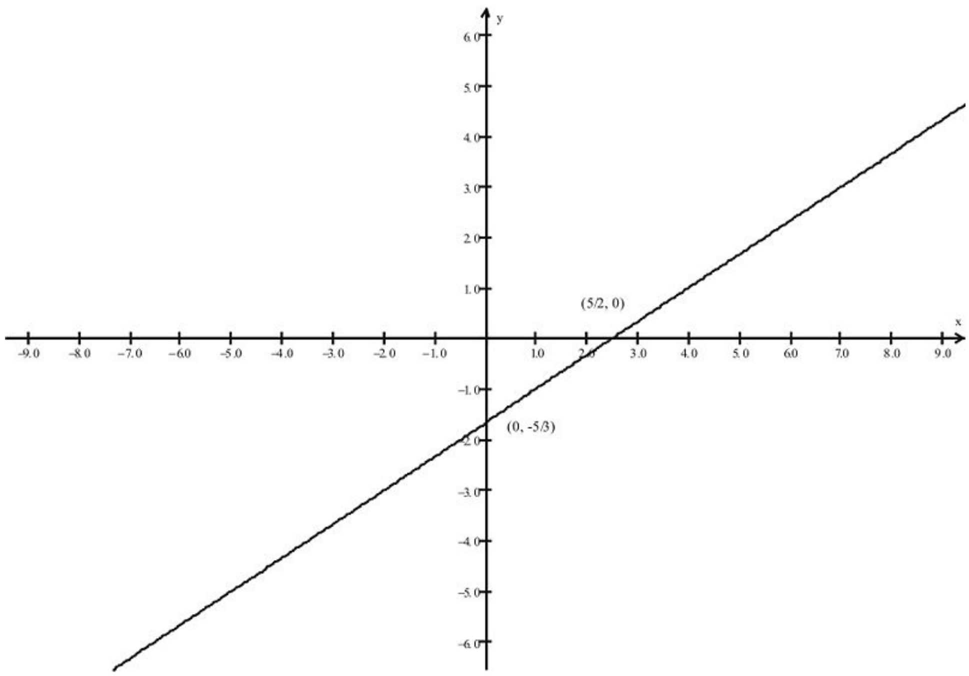
1. Reviewed graphs of equations

2. Reviewed how to find the intercepts of a graph of an equation and to find symmetry in the graph
3. Reviewed how relationships can be used as models of real-life phenomena
4. Reviewed how to solve problems that involve graphs and relationships

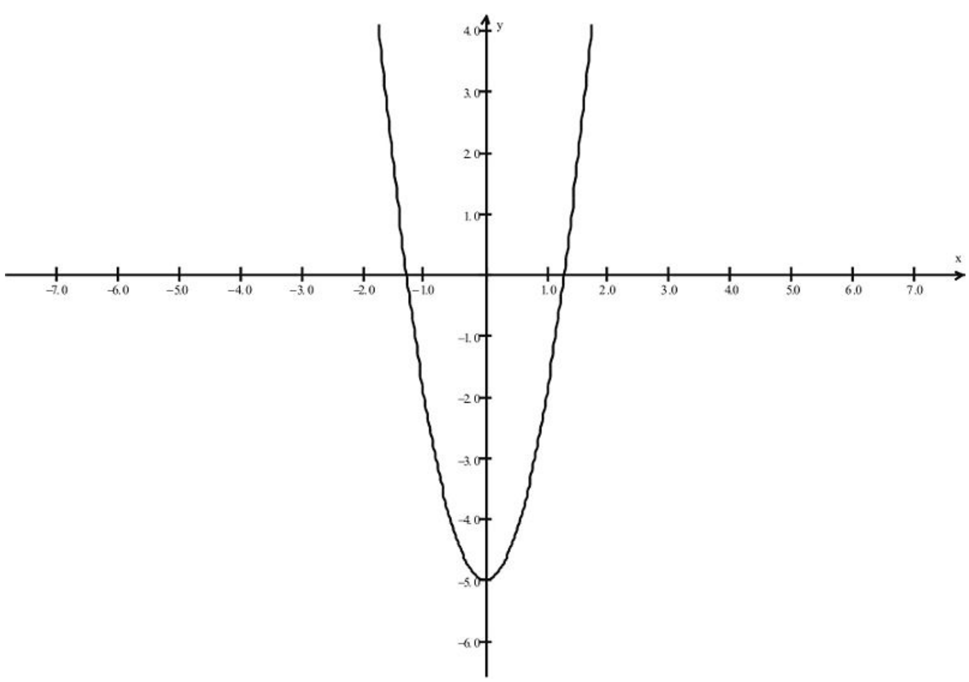
Review Questions

In each of problems 1 - 4, find a pair of solutions of the equation, the intercepts of the graph, and determine if the graph has symmetry.

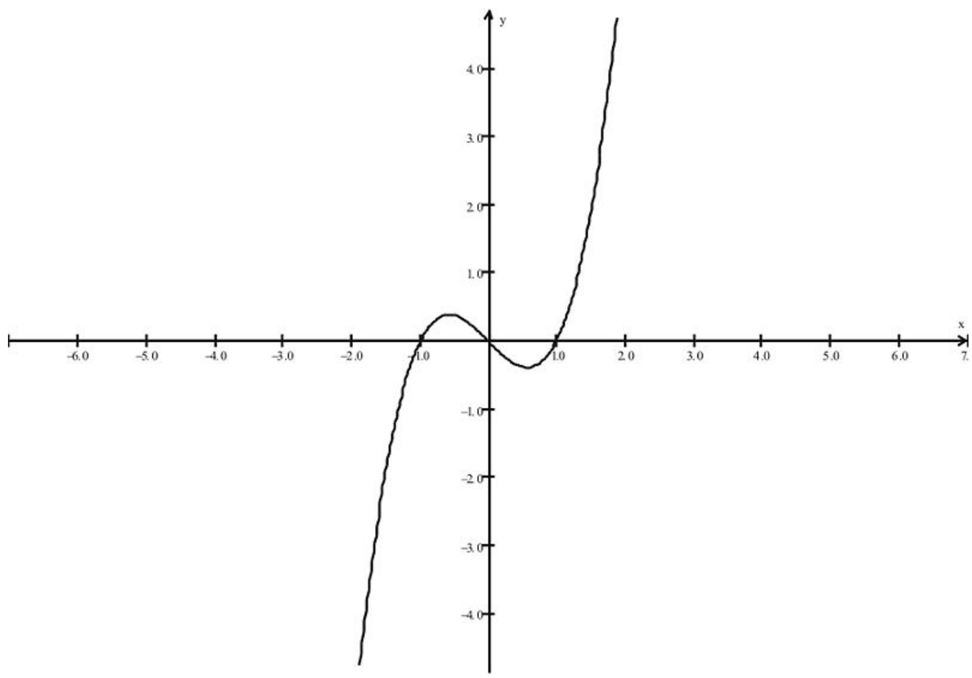
1. $2x-3y=5$



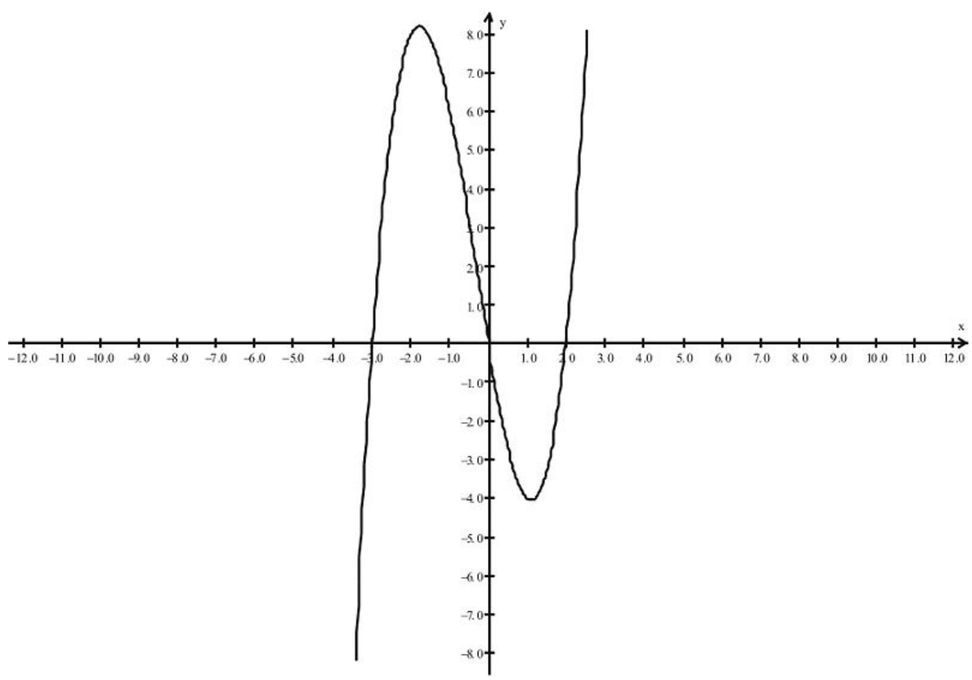
2. $3x^2-y=5$



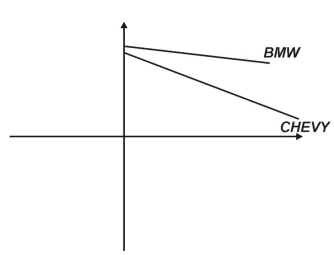
3. $y = x^3 - x$



4. $y = x^3 + x^2 - 6x$

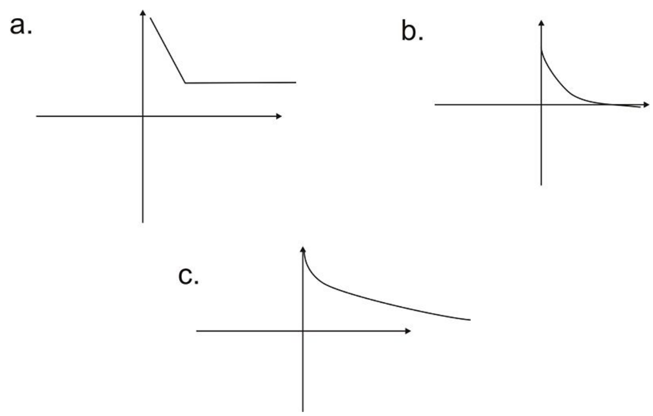


5. Once a car is driven off of the dealership lot, it loses a significant amount of its resale value. The graph below shows the depreciated value of a BMW versus that of a Chevy after t years. Which of the following statements is the best conclusion about the data?



a. You should buy a BMW because they are better cars.
b. BMWs appear to retain their value better than Chevys.
c. The value of each car will eventually be $0.

6. Which of the following graphs is a more realistic representation of the depreciation of cars.



7. A rectangular swimming pool has length that is 25 yards greater than its width.
   a. Give the area enclosed by the pool as a function of its width.
   b. Find the dimensions of the pool if it encloses an area of 264 square yards.
8. Suppose you purchased a car in 2004 for $\$18,000$. You have just found out that the current year 2008 value of your car is $\$8,500$. Assuming that the rate of depreciation of the car is constant, find a formula that shows changing value of the car from 2004 to 2008.
9. For problem #8, in what year will the value of the vehicle be less than $\$1,400$?
10. For problem #8, explain why using a constant rate of change for depreciation may not be the best way to model depreciation.

1.2 Relations and Functions

Learning Objectives

A student will be able to:

- Identify functions from various relationships.
- Review function notation.
- Determine domains and ranges of particular functions.
- Identify key properties of some basic functions.
- Sketch graphs of basic functions.
- Sketch variations of basic functions using transformations.
- Compose functions.

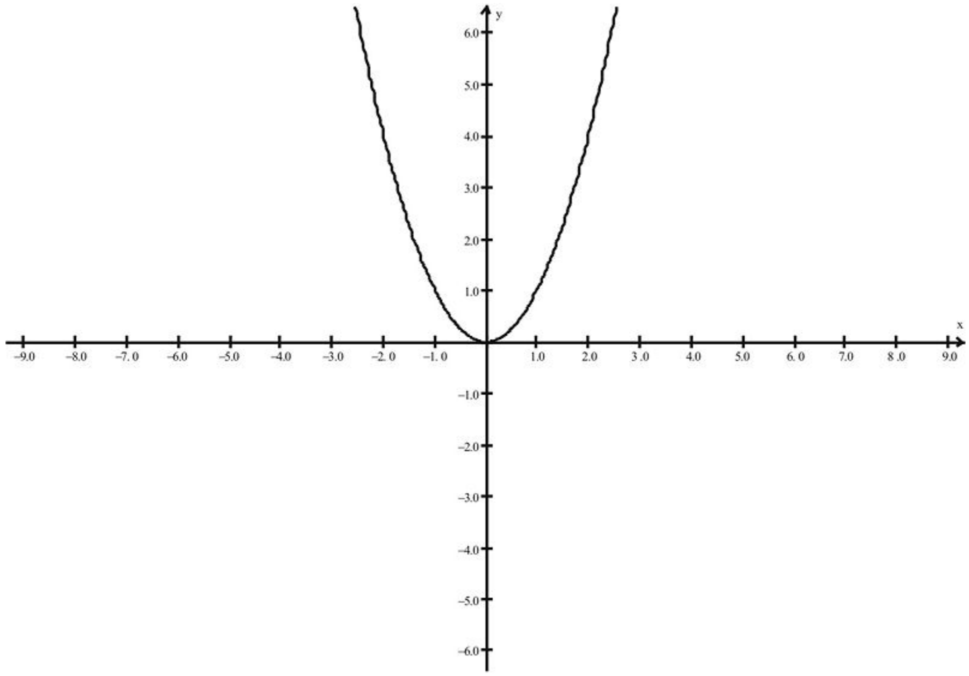
Introduction

In our last lesson we examined a variety of mathematical equations that expressed mathematical relationships. In this lesson we will focus on a particular class of relationships called functions, and examine their key properties. We will then review how to sketch graphs of some basic functions that we will revisit later in this class. Finally, we will examine a way to combine functions that will be important as we develop the key concepts of calculus.

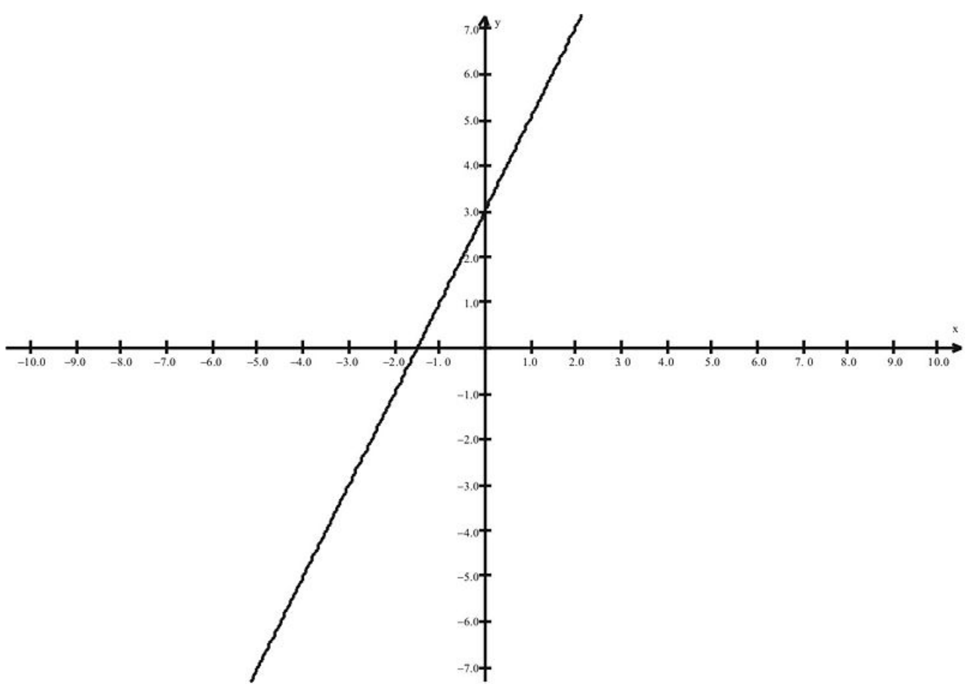
Let's begin our discussion by reviewing four types of equations we examined in our last lesson.

Example 1:

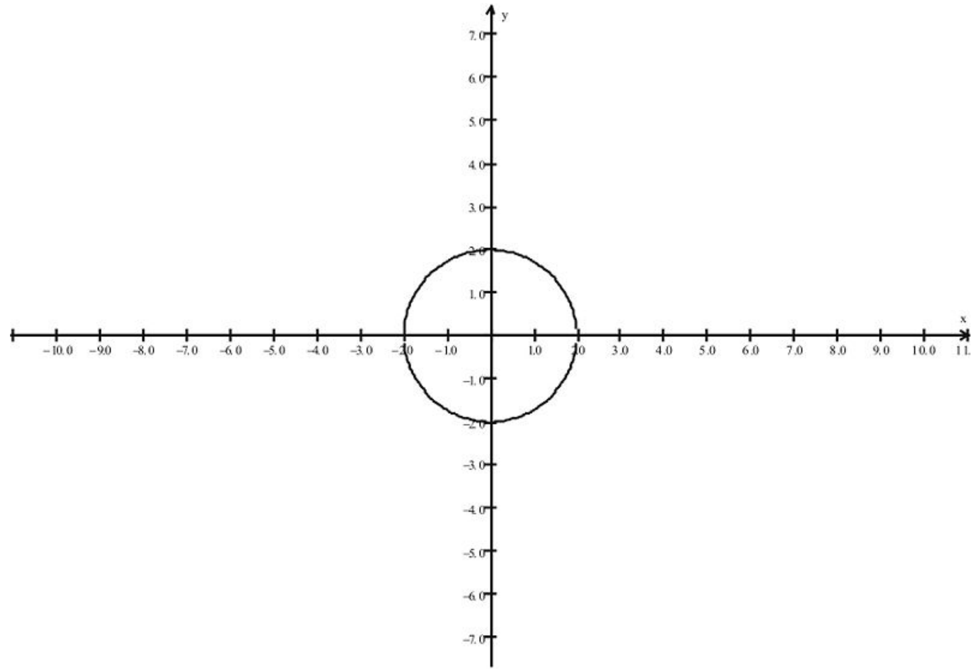
$y = x^2$



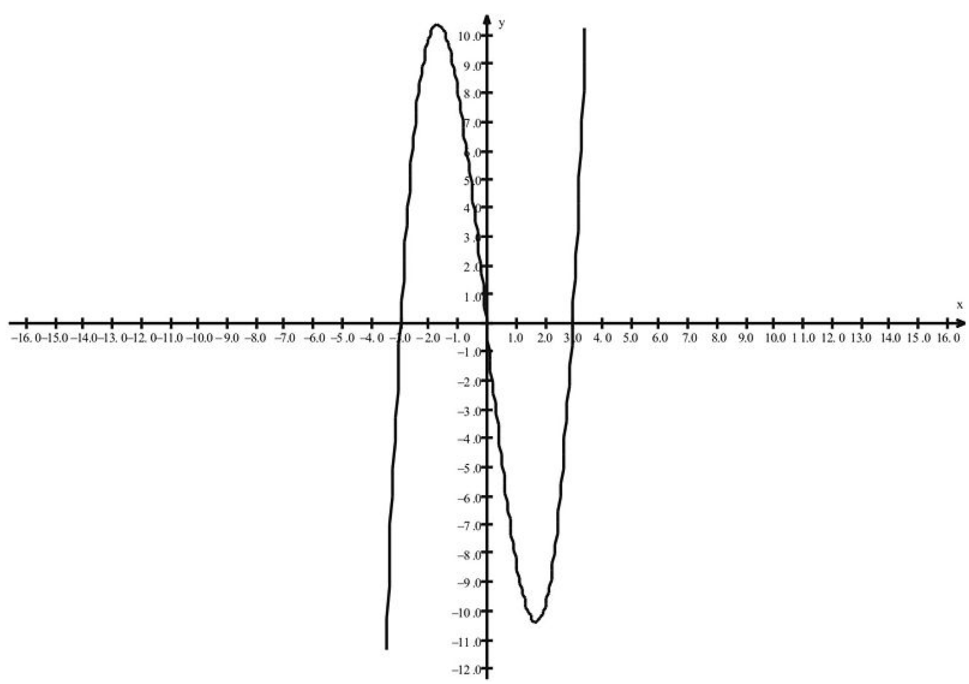
$y = 2x + 3$



$x^2 + y^2 = 4$



$y = x^3 - 9x$

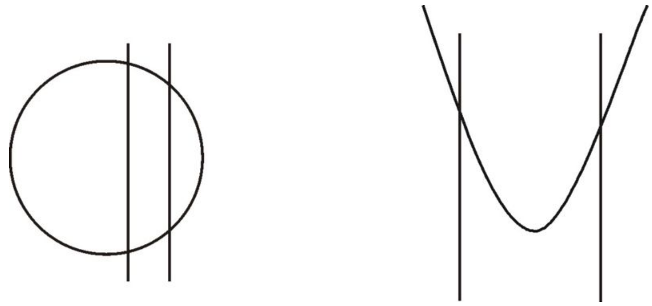


Of these, the circle has a quality that the other graphs do not share. Do you know what it is?

Solution:

The circle's graph includes points where a particular $x-$value has two points associated with it; for example, the points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are both solutions to the equation $x^2 + y^2 = 4$. For each of the other relationships, a particular $x-$value has exactly one $y-$value associated with it.

The relationships that satisfy the condition that for each $x-$value there is a unique $y-$value are called **functions**. Note that we could have determined whether the relationship satisfied this condition by a graphical test, the vertical line test. Recall the relationships of the circle, which is not a function. Let's compare it with the parabola, which is a function.



If we draw vertical lines through the graphs as indicated, we see that the condition of a particular $x-$value having exactly one $y-$value associated with it is equivalent to having at most one point of intersection with any vertical line. The lines on the circle intersect the graph in more than one point, while the lines drawn on the parabola intersect the graph in exactly one point. So this vertical line test is a quick and easy way to check whether or not a graph describes a function.

We want to examine properties of functions such as function notation, their domain and range (the sets of x and y values that define the function), graph sketching techniques, how we can combine functions to get new functions, and also survey some of the basic functions that we will deal with throughout the rest of this book.

Let's start with the notation we use to describe functions. Consider the example of the linear function $y = 2x+3$. We could also describe the function using the symbol $f(x)$ and read as "f *of* x" to indicate the $y-$value of the function for a particular $x-$value. In particular, for this function we would write $f(x) = 2x+3$ and indicate the value of the function at a particular value, say $x = 4$ as $f(4)$ and find its value as follows: $f(4) = 2(4)+3 = 11$. This statement corresponds to the solution $(4,11)$ as a point on the graph of the function. It is read, "f *of* 4 *is* 11."

We can now begin to discuss the properties of functions, starting with the ***domain*** and the ***range*** of a function. The ***domain*** refers to the set of $x-$values that are inputs in the function, while the ***range*** refers to the set of $y-$values that the function takes on. Recall our examples of functions:

Linear Function $g(x) = 2x+3$

Quadratic Function $f(x) = x^2$

Polynomial Function $p(x) = x^3 - 9x$

We first note that we could insert any real number for an $x-$value and a well-defined $y-$value would come out. Hence each function has the set of all real numbers as a domain and we indicate this in interval form as $D : (-\infty,\infty)$. Likewise we see that our graphs could extend up in a positive direction and down in a negative direction without end in either direction. Hence we see that the set of $y-$values, or the range, is the set of all real numbers $R : (-\infty,\infty)$.

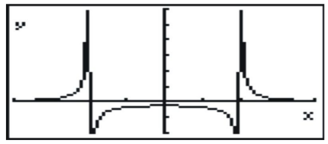
Example 2:

Determine the domain and range of the function.

$$f(x) = 1/(x^2 - 4).$$

Solution:

We note that the condition for each $y-$value is a fraction that includes an x term in the denominator. In deciding what set of $x-$values we can use, we need to exclude those values that make the denominator equal to 0. Why? **(Answer: division by 0 is not defined for real numbers.)** Hence the set of all permissible $x-$values, is all real numbers except for the numbers $(2,-2)$, which yield division by zero. So on our graph we will not see any points that correspond to these $x-$values. It is more difficult to find the range, so let's find it by using the graphing calculator to produce the graph.



From the graph, we see that every $y \neq 0$ value in $(-\infty, \infty)$ (or "All real numbers") is represented; hence the range of the function is $\{-\infty, 0\} \cup \{0, \infty\}$. This is because a fraction with a non-zero numerator never equals zero.

Eight Basic Functions

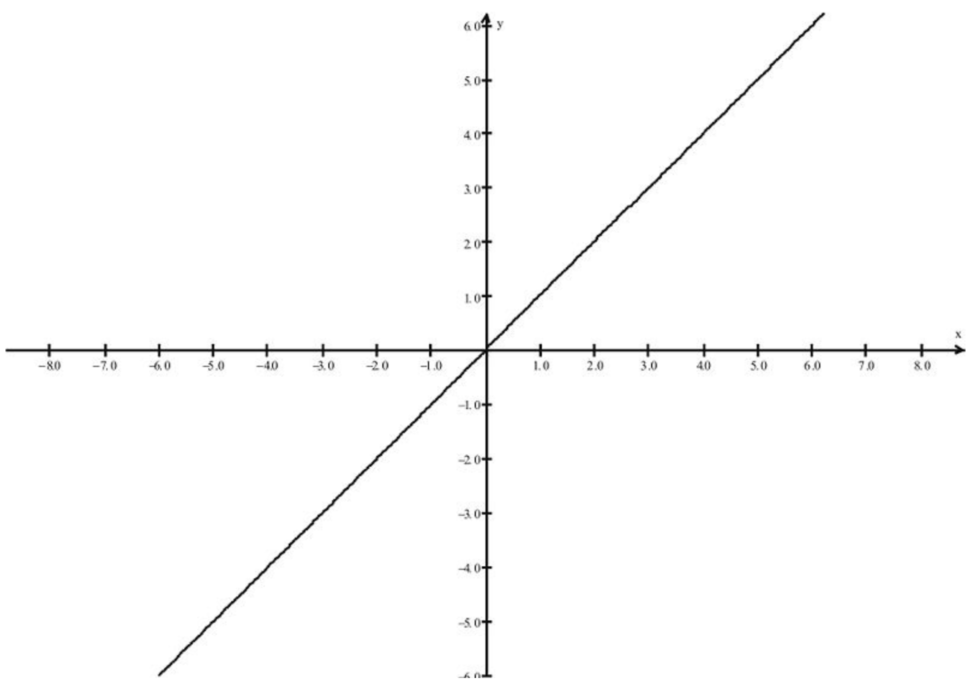
We now present some basic functions that we will work with throughout the course. We will provide a list of eight basic functions with their graphs and domains and ranges. We will then show some techniques that you can use to graph variations of these functions.

Linear

$f(x) = x$

Domain $=$ ***All reals***

Range $=$ ***All reals***

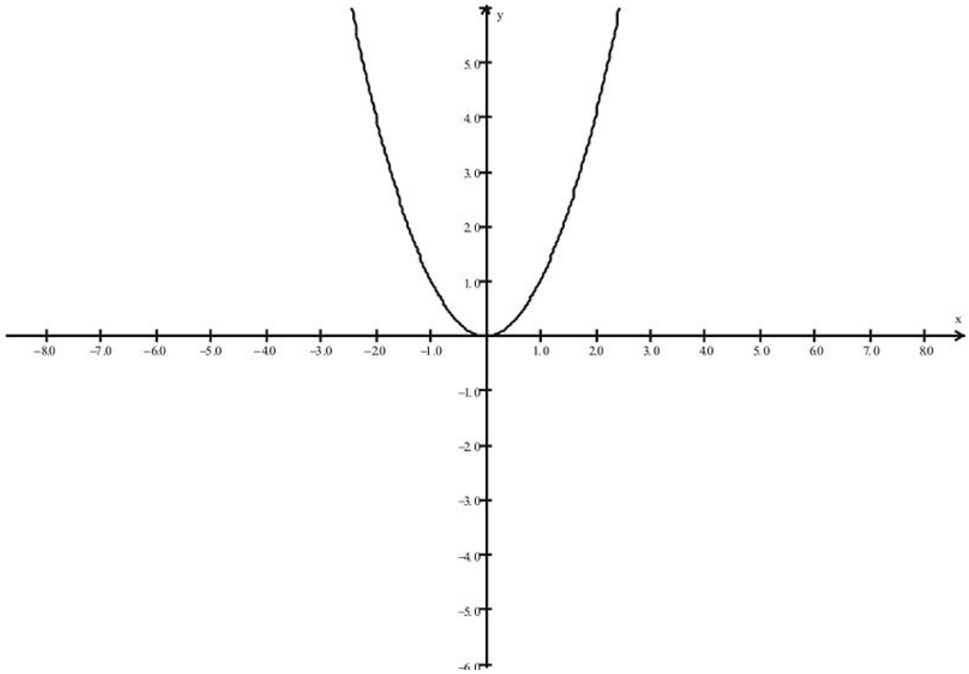


Square (Quadratic)

$f(x) = x^2$

Domain $=$ ***All reals***

Range $= \{y \geq 0\}$

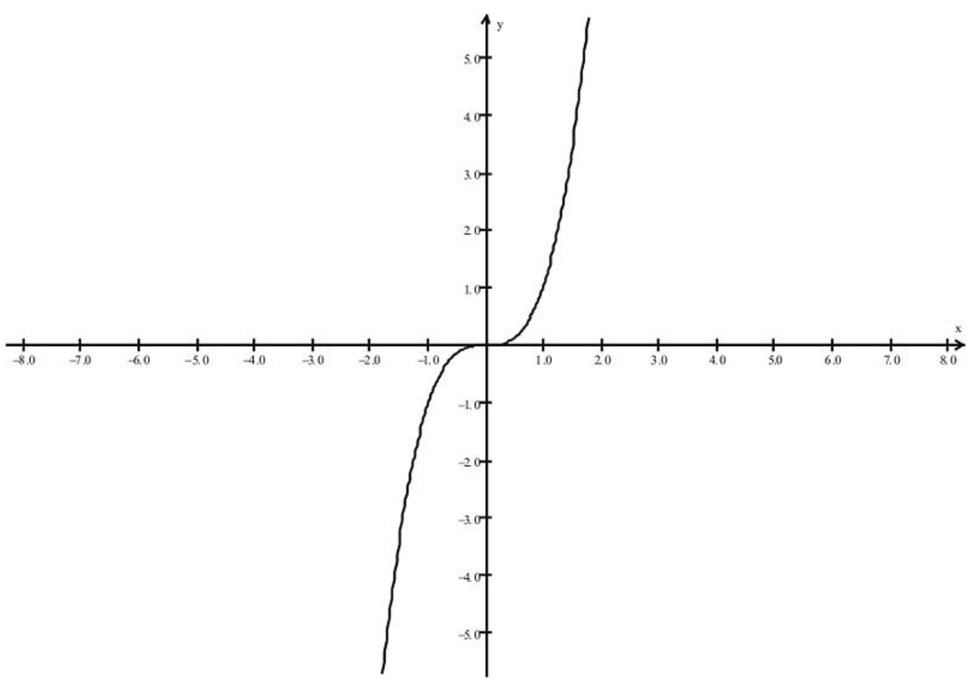


Cube (Polynomial)

$f(x) = x^3$

Domain = All reals

Range = All reals

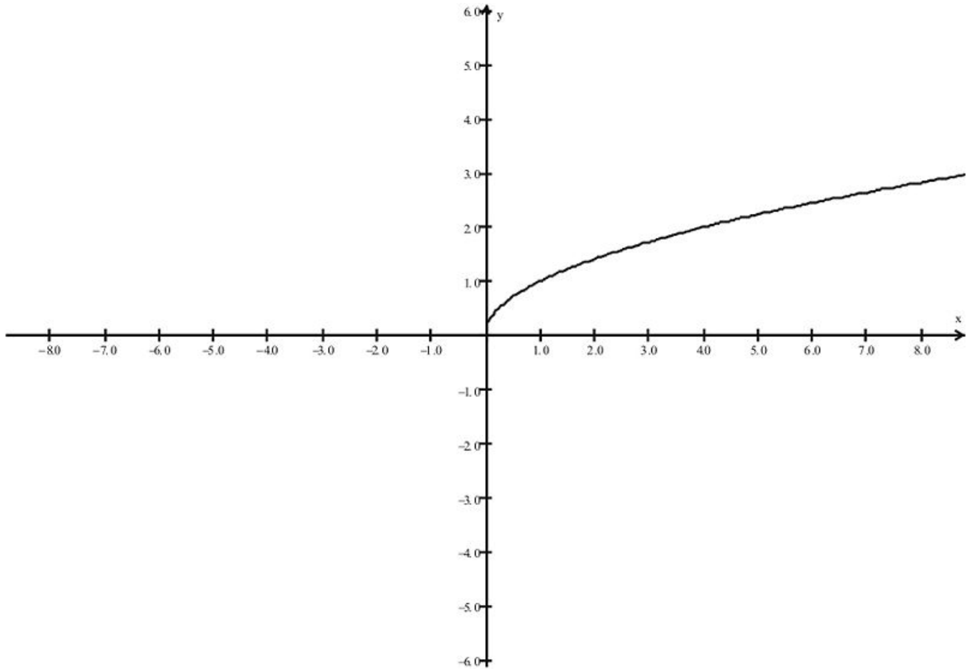


Square Root

$f(x) = \sqrt{x}$

Domain $= \{x \geq 0\}$

Range $= \{y \geq 0\}$

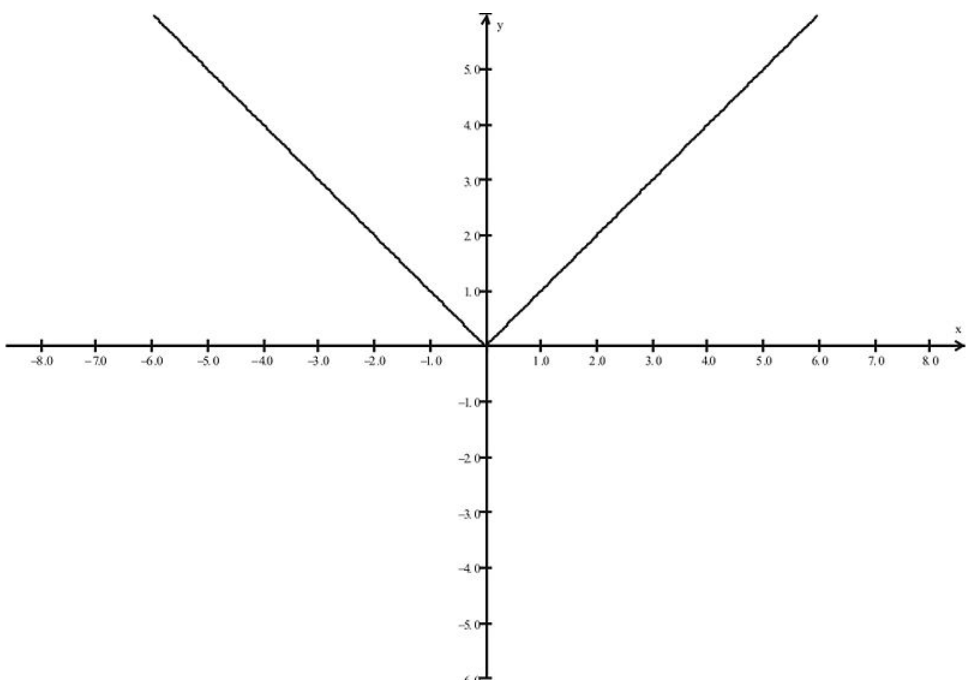


Absolute Value

$f(x) = |x|$

Domain = ***All reals***

Range $= \{y \geq 0\}$

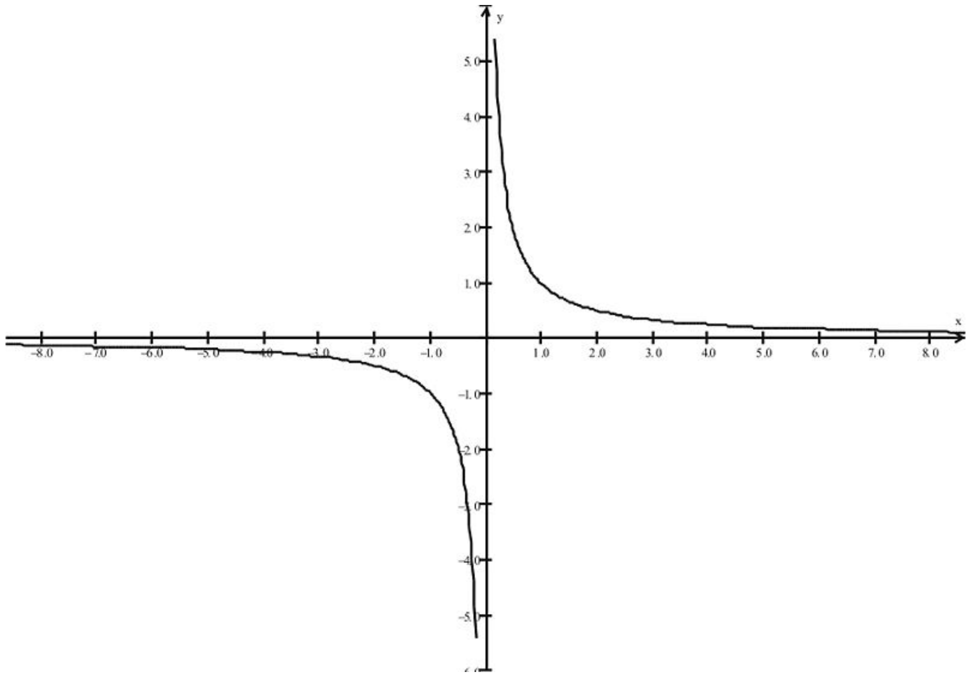


Reciprocal

$f(x) = 1/x$

Domain $= \{x \neq 0\}$

Range $= \{y \neq 0\}$

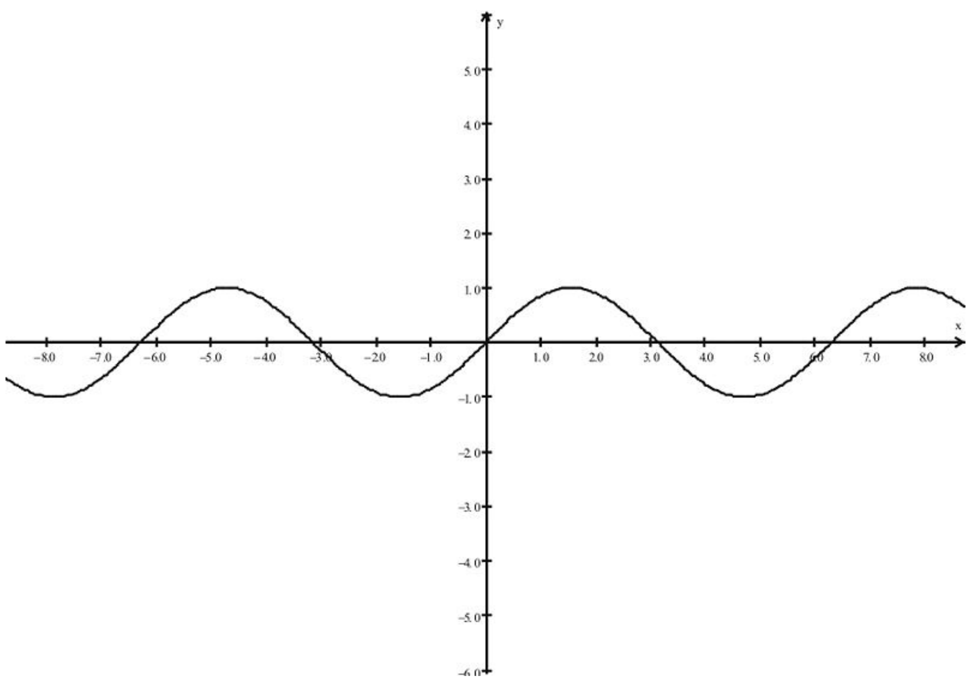


Sine

$f(x) = \sin x$

Domain $=$ ***All reals***

Range $= \{-1 \leq y \leq 1\}$

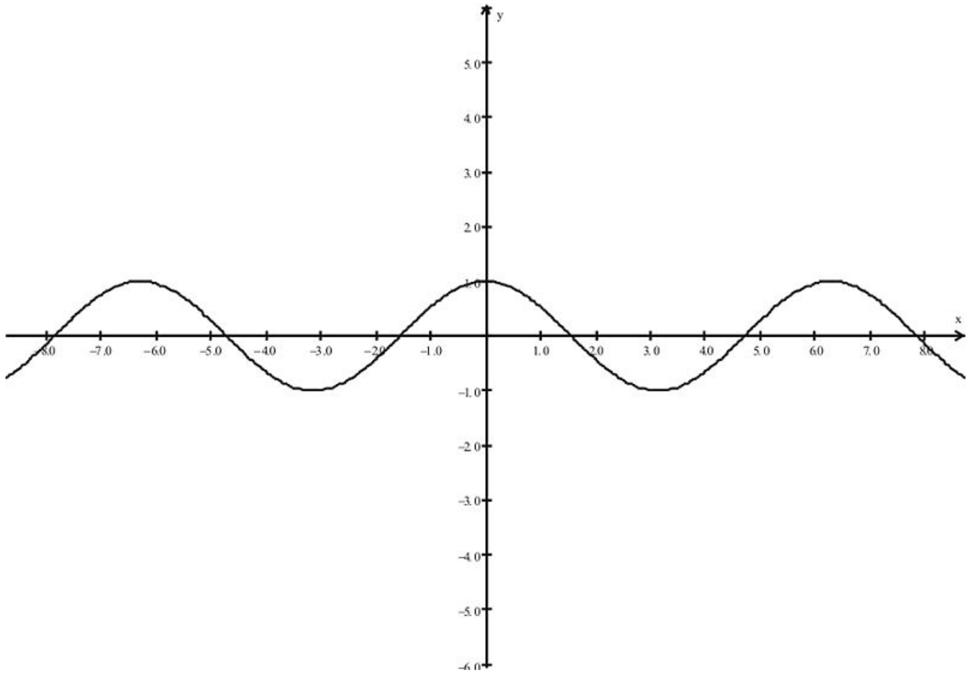


Cosine

$f(x) = \cos x$

Domain $=$ ***All reals***

Range $= \{-1 \leq y \leq 1\}$

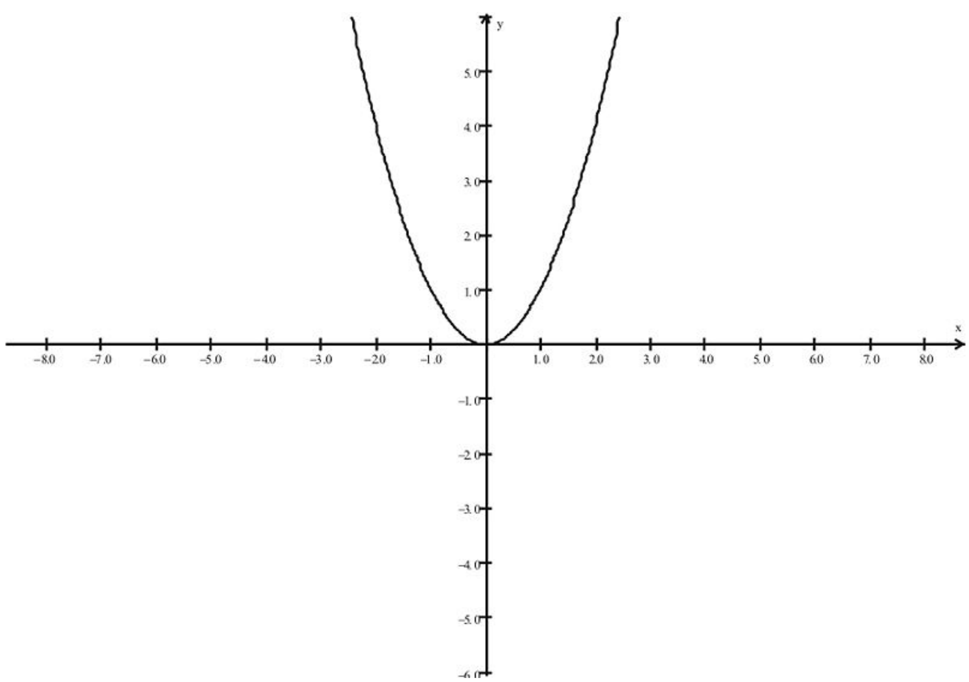


Graphing by Transformations

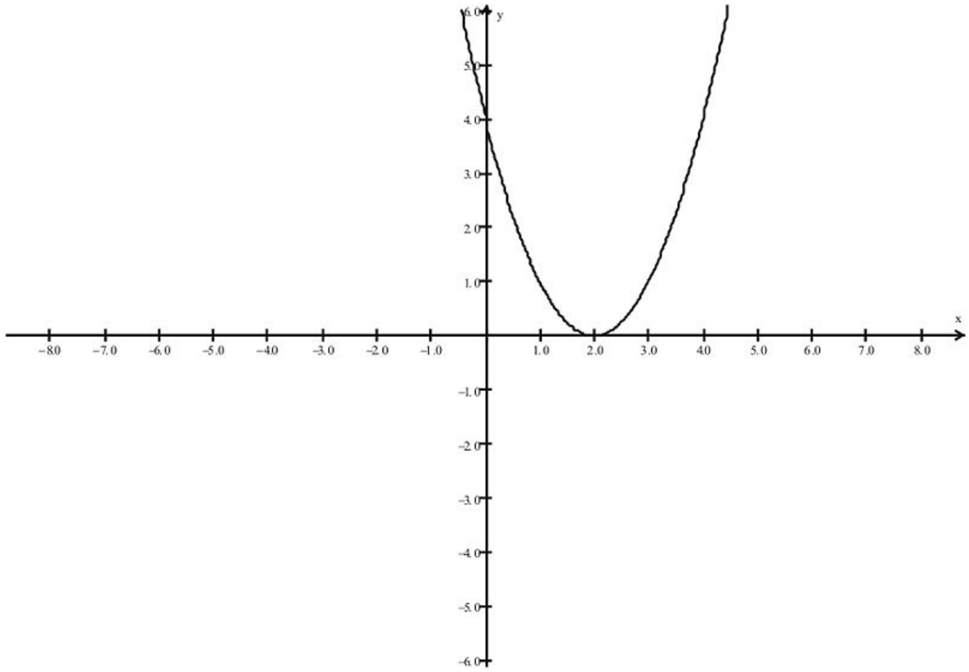
Once we have the basic functions and each graph in our memory, we can easily sketch variations of these. In general, if we have $f(x)$, and c is some constant value, then the graph of $f(x-c)$ is just the graph of $f(x)$ shifted c units to the right. Similarly, the graph of $f(x+c)$ is just the graph of $f(x)$ shifted c units to the left.

Example 3:

$f(x) = x^2$



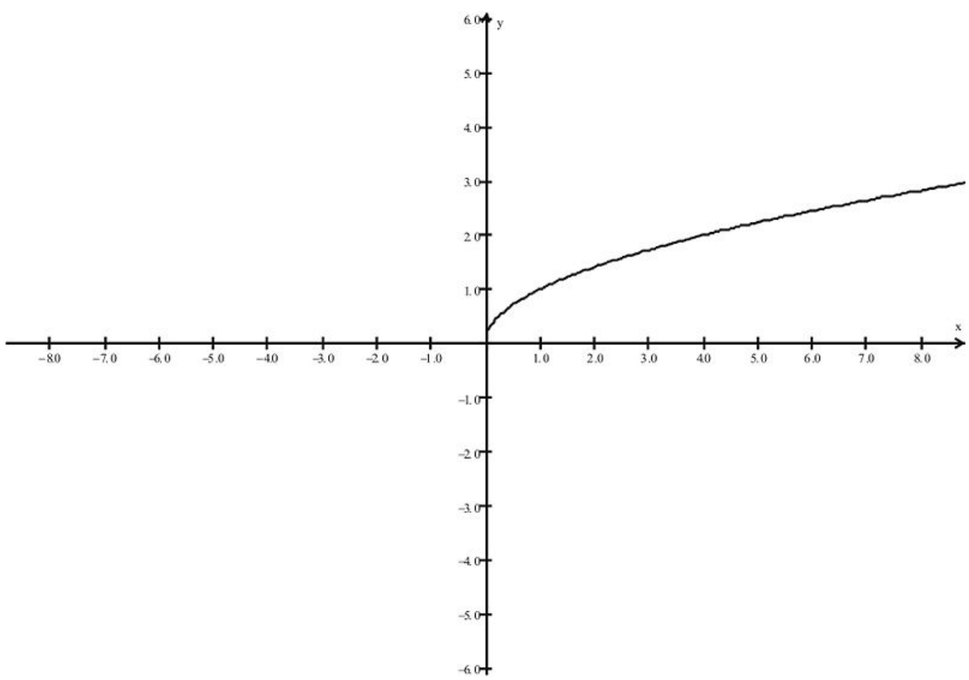
$f(x) = (x-2)^2$



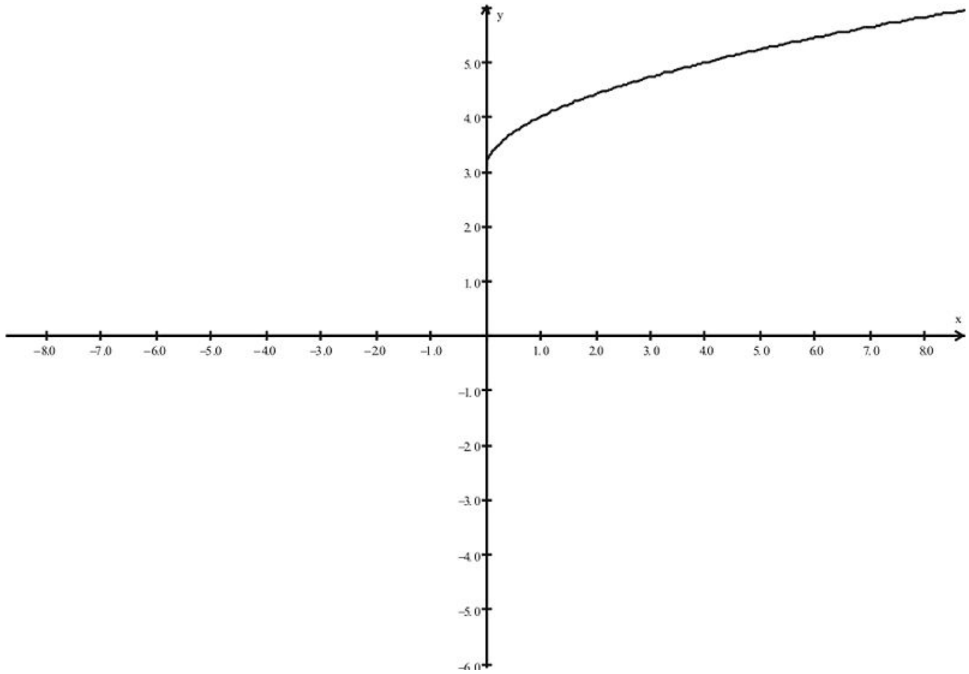
In addition, we can shift graphs up and down. In general, if we have $f(x)$, and c is some constant value, then the graph of $f(x)+c$ is just the graph of $f(x)$ shifted c units up on the $y-$axis. Similarly, the graph of $f(x)-c$ is just the graph of $f(x)$ shifted c units down on the $y-$axis.

Example 4:

$f(x) = \sqrt{x}$

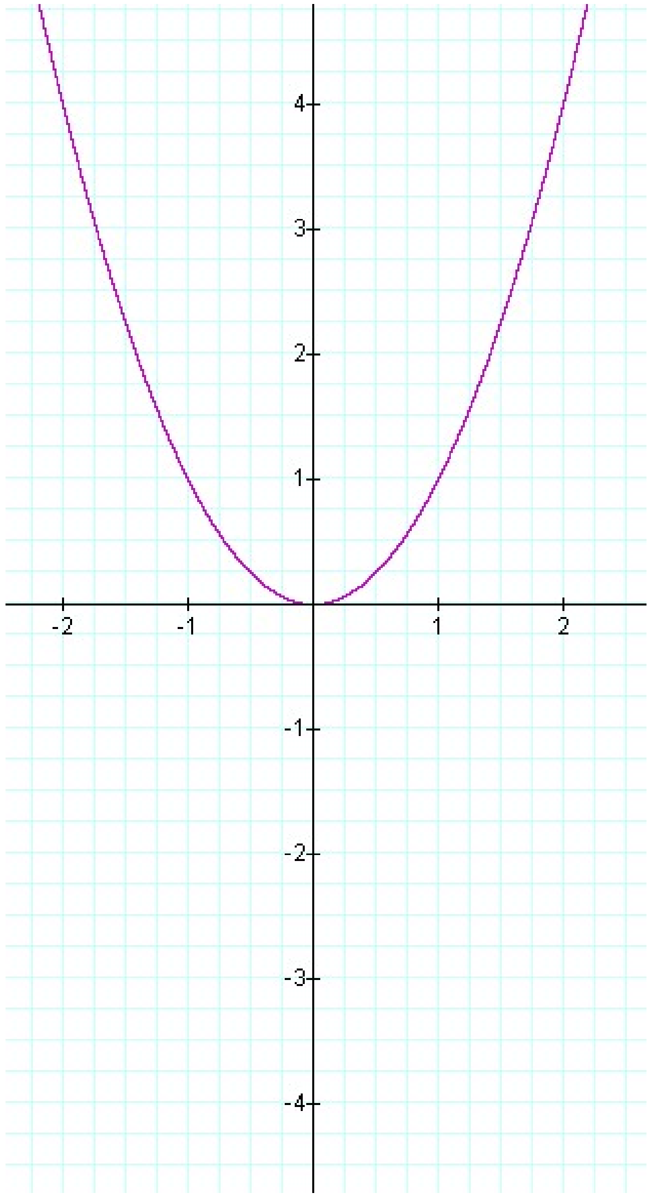


$f(x) = \sqrt{x}+3$

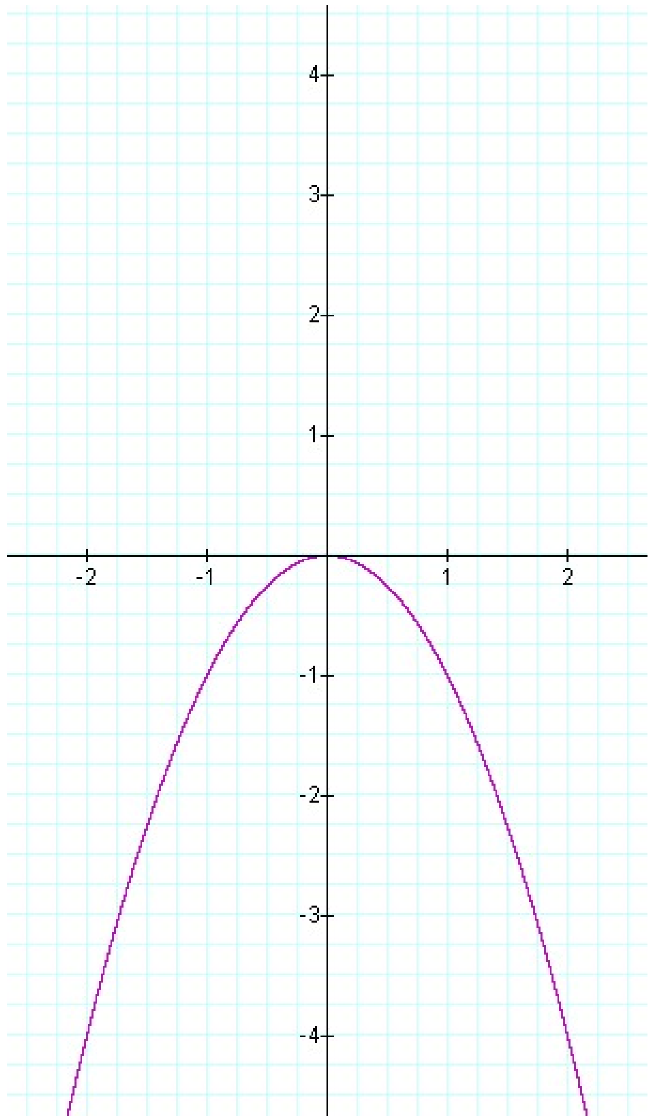


We can also flip graphs in the $x-$axis by multiplying by a negative coefficient.

$f(x) = x^2$



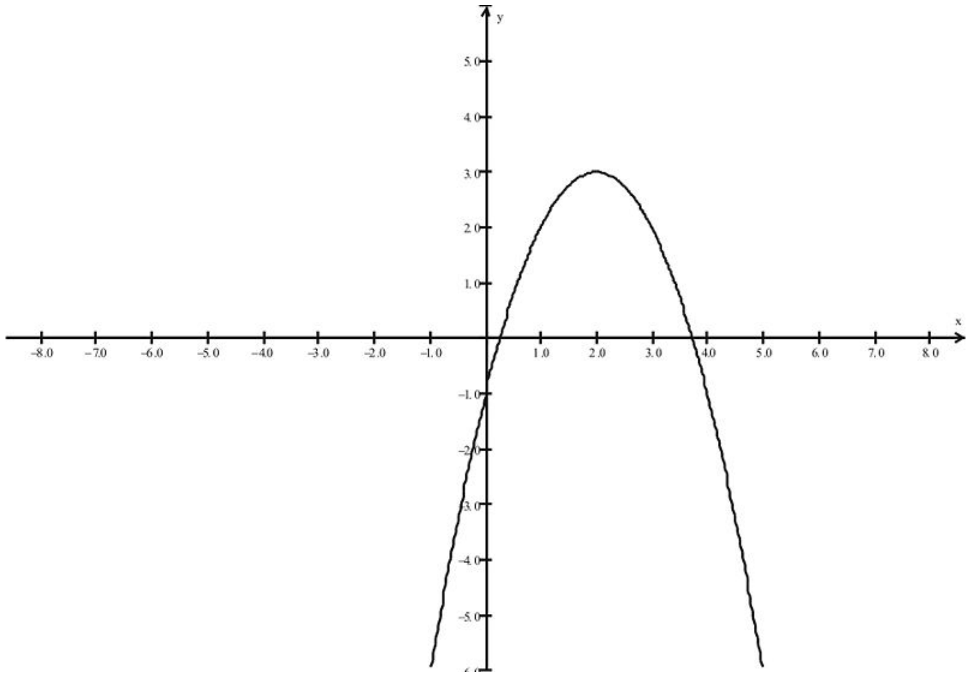
$f(x) = -x^2$



Finally, we can combine these transformations into a single example as follows.

Example 5:

$f(x) = -(x-2)^2 + 3$. The graph will be generated by taking $f(x) = x^2$, flipping in the $y-$axis, and moving it two units to the right and up three units.



Function Composition

The last topic for this lesson involves a way to combine functions called **function composition**. Composition of functions enables us to consider the effects of one function followed by another. Our last example of graphing by transformations provides a nice illustration. We can think of the final graph as the effect of taking the following steps:

$$x \to -(x-2)^2 \to -(x-2)^2+3$$

We can think of it as the application of two functions. First, $g(x)$ takes x to $-(x-2)^2$ and then we apply a second function, $f(x)$ to those $y-$values, with the second function adding $+3$ to each output. We would write the functions as

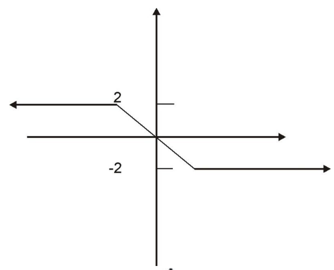
$f(g(x)) = -(x-2)^2+3$ where $g(x) = -(x-2)^2$ and $f(x) = x+3$. We call this operation the composing of f with g and use notation $f \circ g$. Note that in this example, $f \circ g \neq g \circ f$. Verify this fact by computing $g \circ f$ right now. (Note: this fact can be verified algebraically, by showing that the expressions $f \circ g$ and $g \circ f$ differ, or by showing that the different function decompositions are not equal for a specific value.)

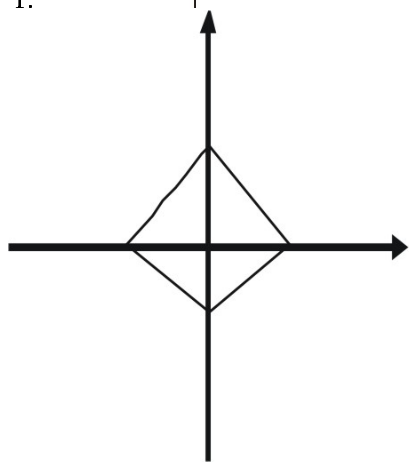
Lesson Summary

1. Learned to identify functions from various relationships.
2. Reviewed the use of function notation.
3. Determined domains and ranges of particular functions.
4. Identified key properties of basic functions.
5. Sketched graphs of basic functions.
6. Sketched variations of basic functions using transformations.
7. Learned to compose functions.

Review Questions

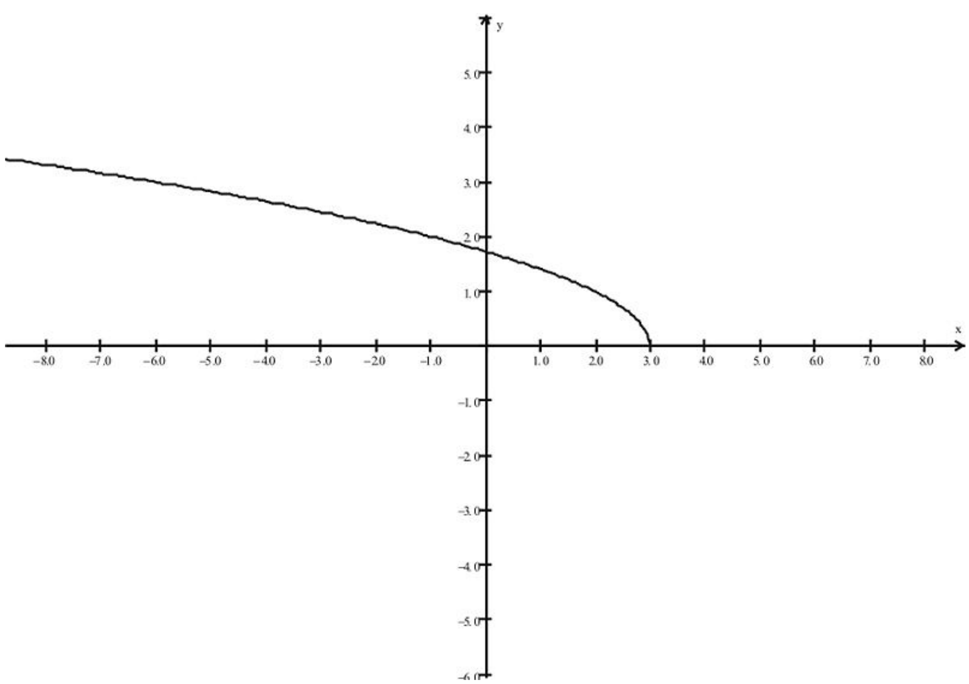
In problems 1 - 2, determine if the relationship is a function. If it is a function, give the domain and range of the function.

1. 

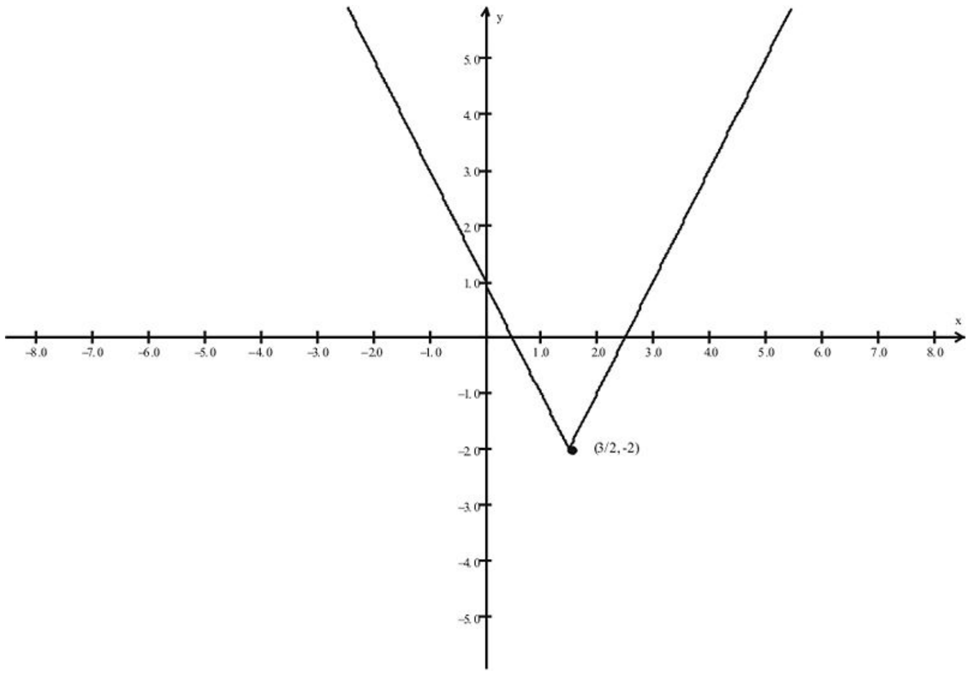
2. 

In problems 3 - 5, determine the domain and range of the function and sketch the graph if no graph is provided.

3. $f(x) = \frac{3x^2}{x^2-1}$
4. $y = \sqrt{-x+3}$



5. $f(x) = |2x-3|-2$



In problems 6 - 8, sketch the graph using transformations of the graphs of basic functions.

6. $f(x) = -(x+2)^2 + 5$
7. $f(x) = -\frac{1}{x-2} + 3$
8. $y = -\sqrt{-x-2} + 3$
9. Find the composites, $f \circ g$ and $g \circ f$ for the following functions.

$$f(x) = -3x + 2, g(x) = \sqrt{x}$$

10. Find the composites, $f \circ g$ and $g \circ f$ for the following functions.

$$f(x) = x^2, g(x) = \sqrt{x}$$

1.3 Models and Data

Learning Objectives

A student will be able to:

- Fit data to linear models.
- Fit data to quadratic models.
- Fit data to trigonometric models.
- Fit data to exponential growth and decay models.

Introduction

In our last lesson we examined functions and learned how to classify and sketch functions. In this lesson we will use some classic functions to model data. The lesson will be a set of examples of each of the models. For each, we will make extensive use of the graphing calculator.

Let's do a quick review of how to model data on the graphing calculator.

Enter Data in Lists

Press **[STAT]** and then **[EDIT]** to access the lists, **L1 - L6**.

View a Scatter Plot

Press **2nd [STAT PLOT]** and choose accordingly.

Then press **[WINDOW]** to set the limits of the axes.

Compute the Regression Equation

Press **[STAT]** then choose **[CALC]** to access the regression equation menu. Choose the appropriate regression equation (Linear, Quad, Cubic, Exponential, Sine).

Graph the Regression Equation Over Your Scatter Plot

Go to **Y=>[MENU]** and clear equations. Press **[VARS]**, then enter 5 and **EQ** and press **[ENTER]** (This series of entries will copy the regression equation to your **Y =** screen.) Press **[GRAPH]** to view the regression equation over your scatter plot

Plotting and Regression in Excel

You can also do regression in an Excel spreadsheet. To start, copy and paste the table of data into Excel. With the two columns highlighted, including the column headings, click on the **Chart** icon and select **XY scatter**. Accept the defaults until a graph appears. Select the graph, then click **Chart**, then **Add Trendline**. From the choices of trendlines choose **Linear**.

Now let's begin our survey of the various modeling situations.

Linear Models

For these kinds of situations, the data will be modeled by the classic linear equation $y = mx + b$. Our task will be to find appropriate values of m and b for given data.

Example 1:

It is said that the height of a person is equal to his or her wingspan (the measurement from fingertip to fingertip when your arms are stretched horizontally). If this is true, we should be able to take a table of measurements, graph the measurements in an $x - y$ coordinate system, and verify this relationship. What kind of graph would you expect to see? ***(Answer: You would expect to see the points on the line** $y = x$.)

Suppose you measure the height and wingspans of nine of your classmates and gather the following data. Use your graphing calculator to see if the following measurements fit this linear model (the line $y = x$).

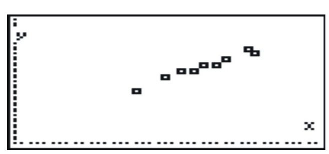
TABLE 1.1:

Height (inches)	*Wingspan (inches)*
67	65
64	63
56	57
60	61
62	63
71	70
72	69
68	67
65	65

We observe that only one of the measurements has the condition that they are equal. Why aren't more of the measurements equal to each other? ***(Answer: The data do not always conform to exact specifications of the model. For example, measurements tend to be loosely documented so there may be an error arising in the way that measurements were taken.)***

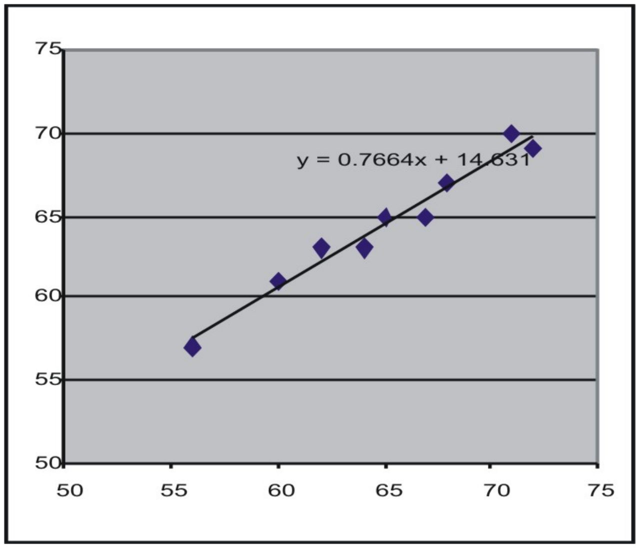
We enter the data in our calculator in **L1** and **L2**. We then view a scatter plot. (Caution: note that the data ranges exceed the viewing window range of $[-10, 10]$. Change the window ranges accordingly to include all of the data, say $[40, 80]$.)

Here is the scatter plot:



Now let us compute the regression equation. Since we expect the data to be linear, we will choose the **linear regression** option from the menu. We get the equation $y = .76x + 14$.

In general we will always wish to graph the regression equation over our data to see the goodness of fit. Doing so yields the following graph, which was drawn with Excel:



Since our calculator will also allow for a variety of non-linear functions to be used as models, we can therefore examine quite a few real life situations. We will first consider an example of quadratic modeling.

Quadratic Models

Example 2:

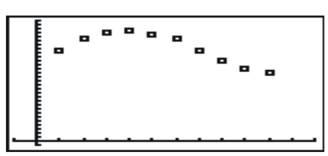
The following table lists the number of Food Stamp recipients (in millions) for each year after 1990.

TABLE 1.2:

years after 1990	*Participants*
1	22.6
2	25.4
3	27.0
4	27.5
5	26.6
6	25.5
7	22.5
8	19.8
9	18.2
10	17.2

We enter the data in our calculator in **L3** and **L4** (that enables us to save the last example's data). We then will view a scatter plot. Change the window ranges accordingly to include all of the data. Use $[-2,10]$ for x and $[-2,30]$ for y.

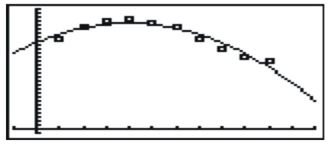
Here is the scatter plot:



Now let us compute the regression equation. Since our scatter plot suggests a quadratic model for the data, we will choose **Quadratic Regression** from the menu. We get the equation:

$$y = -0.30x^2 + 2.38x + 21.67.$$

Let's graph the equation over our data. We see the following graph:



Trigonometric Models

The following example shows how a trigonometric function can be used to model data.

Example 3:

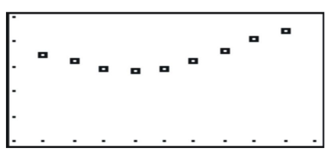
With the skyrocketing cost of gasoline, more people have looked to mass transit as an option for getting around. The following table uses data from the *American Public Transportation Association* to show the number of mass transit trips (in billions) between 1992 and 2000.

TABLE 1.3:

year	*Trips (billions)*
1992	8.5
1993	8.2
1994	7.93
1995	7.8
1996	7.87
1997	8.23
1998	8.6
1999	9.08
2000	9.4

We enter the data in our calculator in **L5** and **L6**, starting in L5 with the number one for 1992 (the first year). We then will view a scatter plot. Change the window ranges accordingly to include all of the data. Use $[-2,10]$ for both x and y ranges.

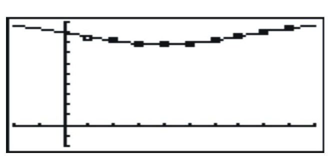
Here is the scatter plot:



Now let us compute the regression equation. Since our scatter plot suggests a sine model for the data, we will choose **Sine Regression** from the menu. We get the equation:

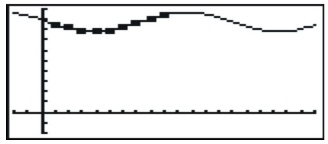
$$y = .9327 * \sin(.4681x + 2.8734) + 8.7358.$$

Let us graph the equation over our data. We see the following graph:



This example suggests that the sine over time t is a function that is used in a variety of modeling situations.

Caution: Although the fit to the data appears quite good, do we really expect the number of trips to continue to go up and down in the future? Probably not. Here is what the graph looks like when projected an additional ten years:



Exponential Models

Our last class of models involves exponential functions. Exponential models can be used to model growth and decay situations. Consider the following data about the declining number of farms for the years 1980 - 2005.

Example 4:

The number of dairy farms has been declining over the past 20+ years. The following table charts the decline:

TABLE 1.4:

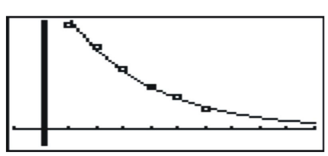
Year	*Farms (thousands)*
1980	334
1985	269
1990	193
1995	140
2000	105
2005	67

We enter the data in our calculator in L5 (entering the year 1980 as 1, the year 1985 as 2, etc.) and L6. We then will view a scatter plot. Change the window ranges accordingly to include all of the data. For the large $y-$values, choose the range $[-50, 350]$ with a scale of 25.

Here is the scatter plot:

Now let us compute the regression equation. Since our scatter plot suggests an exponential model for the data, we will choose **Exponential Regression** from the menu. We get the equation: $y = 490.6317 * .7266^x$

Let's graph the equation over our data. We see the following graph:



In the homework we will practice using our calculator extensively to model data.

Lesson Summary

1. Fit data to linear models.
2. Fit data to quadratic models.
3. Fit data to trigonometric models.
4. Fit data to exponential growth and decay models.

Review Questions

1. Consider the following table of measurements of circular objects:
   a. Make a scatter plot of the data.
   b. Based on your plot, which type of regression will you use?
   c. Find the line of best fit.
   d. Comment on the values of m and b in the equation.

TABLE 1.5:

Object	*Diameter (cm)*	*Circumference (cm)*
Glass	8.3	26.5
Flashlight	5.2	16.7
Aztec calendar	20.2	61.6
Tylenol bottle	3.4	11.6
Popcorn can	13	41.4
Salt shaker	6.3	20.1
Coffee canister	11.3	35.8
Cat food bucket	33.5	106.5
Dinner plate	27.3	85.6
Ritz cracker	4.9	15.5

2. Manatees are large, gentle sea creatures that live along the Florida coast. Many manatees are killed or injured by power boats. Here are data on powerboat registrations (in thousands) and the number of manatees killed by boats in Florida from 1987 - 1997.
   a. Make a scatter plot of the data.
   b. Use linear regression to find the line of best fit.
   c. Suppose in the year 2000, powerboat registrations increase to $700,000$. Predict how many manatees will be killed.

TABLE 1.6:

Year	*Boats*	*Manatees killed*
1987	447	13
1988	460	21
1989	480	24
1990	497	16
1991	512	24
1992	513	21

TABLE 1.6: (continued)

Year	*Boats*	*Manatees killed*
1993	526	15
1994	557	33
1995	585	34
1996	614	34
1997	645	39

3. A passage in ***Gulliver's Travels*** states that the measurement of "Twice around the wrist is once around the neck." The table below contains the wrist and neck measurements of 10 people.
    a. Make a scatter plot of the data.
    b. Find the line of best fit and comment on the accuracy of the quote from the book.
    c. Predict the distance around the neck of Gulliver if the distance around his wrist is found to be 52 cm.

TABLE 1.7:

Wrist (cm)	*Neck (cm)*
17.9	39.5
16	32.5
16.5	34.7
15.9	32
17	33.3
17.3	32.6
16.8	33
17.3	31.6
17.7	35
16.9	34

4. The following table gives women's average percentage of men's salaries for the same jobs for each 5-year period from 1960 - 2005.
    a. Make a scatter plot of the data.
    b. Based on your sketch, should you use a linear or quadratic model for the data?
    c. Find a model for the data.
    d. Can you explain why the data seems to dip at first and then grow?

TABLE 1.8:

Year	*Percentage*
1960	42
1965	36
1970	30
1975	37
1980	41
1985	42
1990	48
1995	55
2000	58
2005	60

5. Based on the model for the previous problem, when will women make as much as men? Is your answer a realistic prediction?
6. The average price of a gallon of gas for selected years from 1975 - 2008 is given in the following table:
   a. Make a scatter plot of the data.
   b. Based on your sketch, should you use a linear, quadratic, or cubic model for the data?
   c. Find a model for the data.
   d. If gas continues to rise at this rate, predict the price of gas in the year 2012.

TABLE 1.9:

Year	*Cost*
1975	1
1976	1.75
1981	2
1985	2.57
1995	2.45
2005	2.75
2008	3.45

7. For the previous problem, use a linear model to analyze the situation. Does the linear method provide a better estimate for the predicted cost for the year 2012? Why or why not?
8. Suppose that you place $\$1,000$ in a bank account where it grows exponentially and is compounded annually over the course of six years. The table below shows the amount of money you have at the end of each year.
   a. Find the exponential model.
   b. In what year will you triple your original amount?

TABLE 1.10:

Year	*Amount*
0	1000
1	1127.50
2	1271.24
3	1433.33
4	1616.07
5	1822.11
6	2054.43

9. Suppose that in the previous problem, you started with $\$3,000$ but maintained the same interest rate.
   a. Give a formula for the exponential model. (Hint: note the coefficient in the previous answer!)
   b. How long will it take for the initial amount, $\$3,000$, to triple? Explain your answer.
10. The following table gives the average daily temperature for Indianapolis, Indiana for each month of the year:
   a. Construct a scatter plot of the data.
   b. Find the sine model for the data.

TABLE 1.11:

Month	*Avg Temp (F)*
Jan	22

TABLE 1.11: (continued)

Month	*Avg Temp (F)*
Feb	26.3
March	37.8
April	51
May	61.7
June	75.3
July	78.5
Aug	84.3
Sept	68.5
Oct	53.2
Nov	38.7
Dec	26.6

1.4 The Calculus

Learning Objectives

A student will be able to:

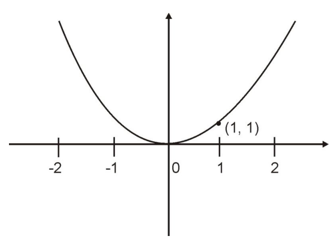
- Use linear approximations to study the limit process.
- Compute approximations for the slope of tangent lines to a graph.
- Introduce applications of differential calculus.

Introduction

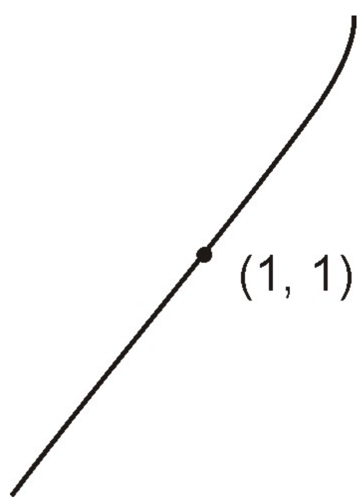
In this lesson we will begin our discussion of the key concepts of calculus. They involve a couple of basic situations that we will come back to time and again throughout the book. For each of these, we will make use of some basic ideas about how we can use straight lines to help approximate functions.

Let's start with an example of a simple function to illustrate each of the situations.

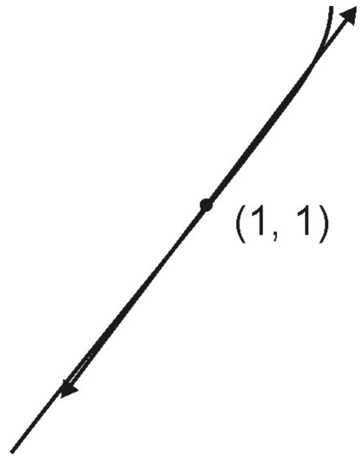
Consider the quadratic function $f(x) = x^2$. We recall that its graph is a parabola. Let's look at the point $(1,1)$ on the graph.



Suppose we magnify our picture and zoom in on the point $(1,1)$. The picture might look like this:



We note that the curve now looks very much like a straight line. If we were to overlay this view with a straight line that intersects the curve at $(1,1)$, our picture would look like this:



We can make the following observations. First, this line would appear to provide a good estimate of the value of $f(x)$ for $x-$values very close to $x = 1$. Second, the approximations appear to be getting closer and closer to the actual vale of the function as we take points on the line closer and closer to the point $(1,1)$. This line is called ***the tangent line to*** $f(x)$ ***at*** $(1,1)$. This is one of the basic situations that we will explore in calculus.

Tangent Line to a Graph

Continuing our discussion of the tangent line to $f(x)$ at $(1,1)$, we next wish to find the equation of the tangent line. We know that it passes though $(1,1)$, but we do not yet have enough information to generate its equation. What other information do we need? ***(Answer: The slope of the line.)***

Yes, we need to find the slope of the line. We would be able to find the slope if we knew a second point on the line. So let's choose a point P on the line, very close to $(1,1)$. We can approximate the coordinates of P using the function $f(x) = x^2$; hence $P(x,x^2)$. Recall that for points very close to $(1,1)$, the points on the line are close approximate points of the function. Using this approximation, we can compute the slope of the tangent as follows:

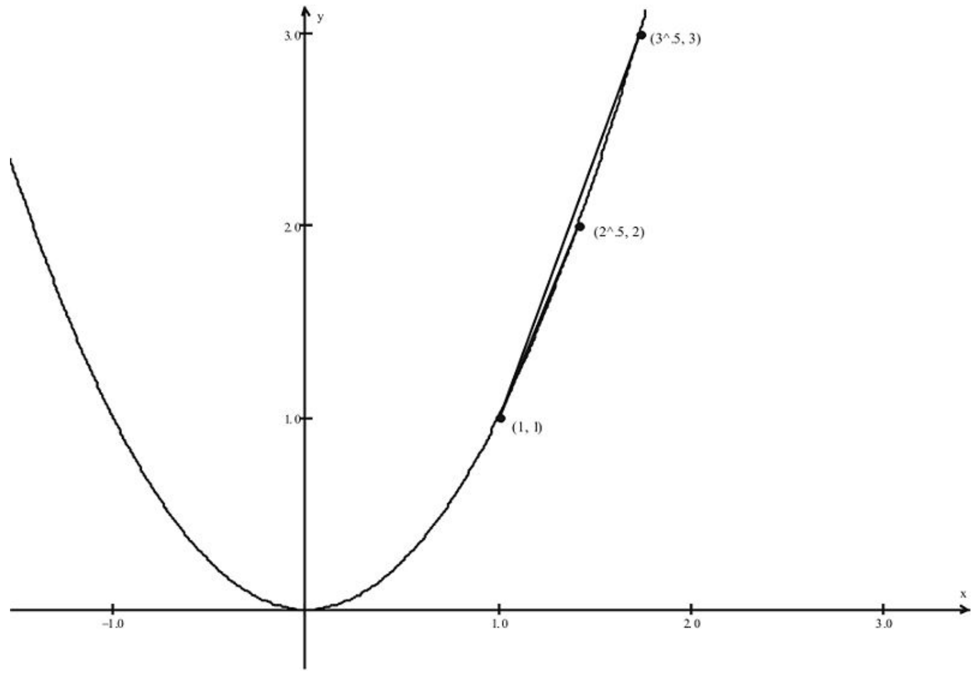
$m = (x^2 - 1)/(x - 1) = x + 1$ (Note: We choose points very close to $(1,1)$ but not the point itself, so $x \neq 1$).

In particular, for $x = 1.25$ we have $P(1.25, 1.5625)$ and $m = x + 1 = 2.25$. Hence the equation of the tangent line, in point slope form is $y - 1 = 2.25(x - 1)$. We can keep getting closer to the actual value of the slope by taking P closer to $(1,1)$, or x closer and closer to $x = 1$, as in the following table:

$P(x,y)$	m
$(1.2, 1.44)$	2.2
$(1.15, 1.3225)$	2.15
$(1.1, 1.21)$	2.1
$(1.05, 1.1025)$	2.05
$(1.005, 1.010025)$	2.005
$(1.0001, 1.00020001)$	2.0001

As we get closer to $(1,1)$, we get closer to the actual slope of the tangent line, the value 2. We call the slope of the tangent line at the point $(1,1)$ **the derivative of the function** $f(x)$ **at the point** $(1,1)$.

Let's make a couple of observations about this process. First, we can interpret the process graphically as finding secant lines from $(1,1)$ to other points on the graph. From the diagram we see a sequence of these secant lines and can observe how they begin to approximate the tangent line to the graph at $(1,1)$. The diagram shows a pair of secant lines, joining $(1,1)$ with points $(\sqrt{2},2)$ and $(\sqrt{3},3)$.

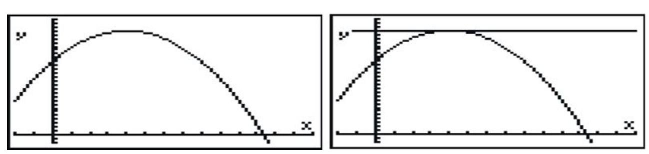


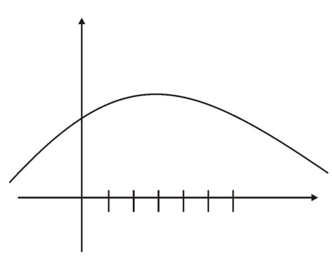
Second, in examining the sequence of slopes of these secants, we are systematically observing ***approximate slopes of the function*** as point P gets closer to $(1,1)$. Finally, producing the table of slope values above was an inductive process in which we generated some data and then looked to deduce from our data the value to which the generated results tended. In this example, the slope values appear to approach the value 2. This process of finding how function values behave as we systematically get closer and closer to particular $x-$values is the process of finding ***limits***. In the next lesson we will formally define this process and develop some efficient ways for computing limits of functions.

Applications of Differential Calculus

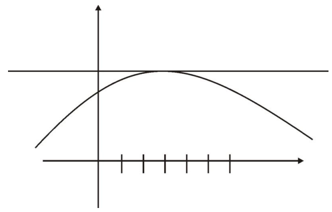
Maximizing and Minimizing Functions

Recall from Lesson 1.3 our example of modeling the number of Food Stamp recipients. The model was found to be $y = -0.5x^2 + 4x + 19$ with graph as follows: (Use viewing window ranges of $[-2,14]$ on x and $[-2,30]$ on y)





We note that the function appears to attain a maximum value about an $x-$value somewhere around $x = 4$. Using the process from the previous example, what can we say about the tangent line to the graph for that x value that yields the maximum y value (the point at the top of the parabola)? ***(Answer: the tangent line will be horizontal, thus having a slope of*** **0*.)***



Hence we can use calculus to model situations where we wish to maximize or minimize a particular function. This process will be particularly important for looking at situations from business and industry where polynomial functions provide accurate models.

Velocity of a Falling Object

We can use differential calculus to investigate the velocity of a falling object. Galileo found that the distance traveled by a falling object was proportional to the square of the time it has been falling:

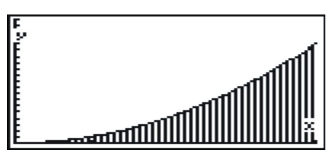
$$s(t) = 4.9t^2.$$

The average velocity of a falling object from $t = a$ to $t = b$ is given by $(s(b) - s(a))/(b - a)$.

HW Problem #10 will give you an opportunity to explore this relationship. In our discussion, we saw how the study of tangent lines to functions yields rich information about functions. We now consider the second situation that arises in Calculus, the central problem of ***finding the area under the curve of a function*** $f(x)$.

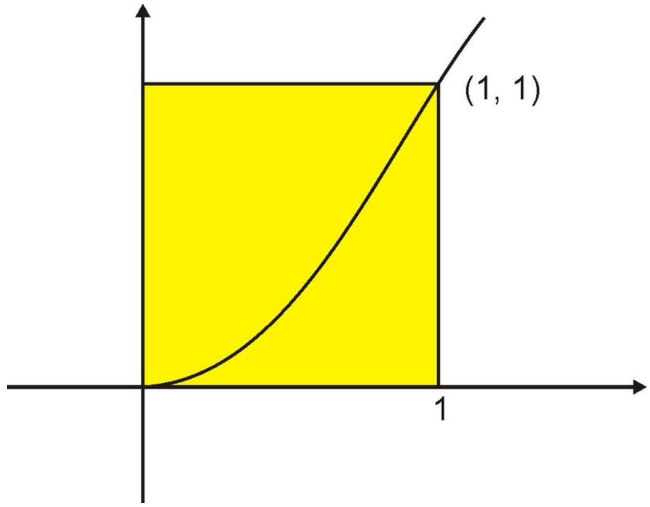
Area Under a Curve

First let's describe what we mean when we refer to the area under a curve. Let's reconsider our basic quadratic function $f(x) = x^2$. Suppose we are interested in finding the area under the curve from $x = 0$ to $x = 1$.

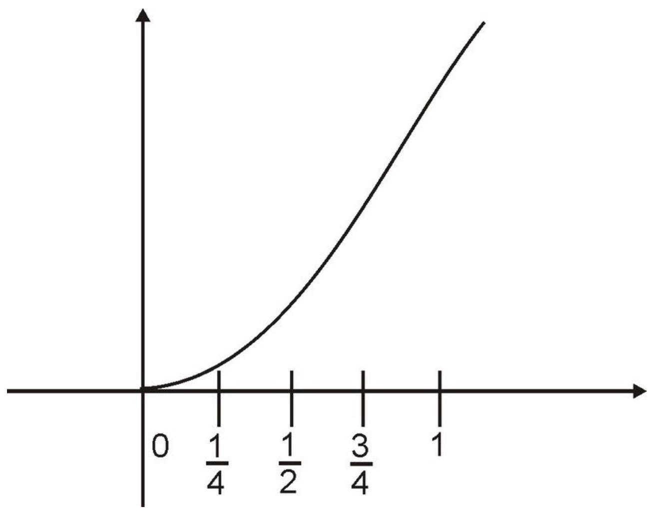


We see the cross-hatched region that lies between the graph and the $x-$axis. That is the area we wish to compute. As with approximating the slope of the tangent line to a function, we will use familiar linear methods to approximate the area. Then we will repeat the iterative process of finding better and better approximations.

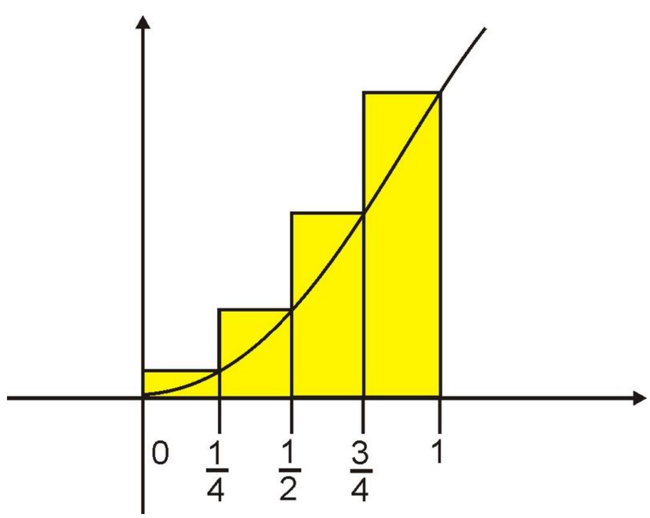
Can you think of any ways that you would be able to approximate the area? (Answer: One ideas is that we could compute the area of the square that has a corner at $(1,1)$ to be $A = 1$ and then take half to find an area $A = 1/2$. This is one estimate of the area and it is actually a pretty good first approximation.)



We will use a variation of this covering of the region with quadrilaterals to get better approximations. We will do so by dividing the $x-$interval from $x = 0$ to $x = 1$ into equal sub-intervals. Let's start by using four such subintervals as indicated:



We now will construct four rectangles that will serve as the basis for our approximation of the area. The subintervals will serve as the width of the rectangles. We will take the length of each rectangle to be the maximum value of the function in the subinterval. Hence we get the following figure:



If we call the rectangles R1–R4, from left to right, then we have the areas

$$R_1 = \frac{1}{4} * f\left(\frac{1}{4}\right) = \frac{1}{64},$$
$$R_2 = \frac{1}{4} * f\left(\frac{1}{2}\right) = \frac{1}{16},$$
$$R_3 = \frac{1}{4} * f\left(\frac{3}{4}\right) = \frac{9}{64},$$
$$R_4 = \frac{1}{4} * f(1) = \frac{1}{4},$$

and $R_1 + R_2 + R_3 + R_4 = \frac{30}{64} = \frac{15}{32}$.

Note that this approximation is very close to our initial approximation of $1/2$. However, since we took the maximum value of the function for a side of each rectangle, this process tends to overestimate the true value. We could have used the minimum value of the function in each sub-interval. Or we could have used the value of the function at the midpoint of each sub-interval.

Can you see how we are going to improve our approximation using successive iterations like we did to approximate the slope of the tangent line? ***(Answer: we will sub-divide the interval from*** $x = 0$ ***to*** $x = 1$ ***into more and more sub-intervals, thus creating successively smaller and smaller rectangles to refine our estimates.)***

Example 1:

The following table shows the areas of the rectangles and their sum for rectangles having width $w = 1/8$.

TABLE 1.12:

Rectangle R_i	Area of R_i
R_1	$\frac{1}{512}$
R_2	$\frac{4}{512}$
R_3	$\frac{9}{512}$
R_4	$\frac{16}{512}$
R_5	$\frac{25}{512}$
R_6	$\frac{36}{512}$
R_7	$\frac{49}{512}$
R_8	$\frac{64}{512}$

$A = \sum R_i = \frac{195}{512}$. This value is approximately equal to .3803. Hence, the approximation is now quite a bit less than .5. For sixteen rectangles, the value is $\frac{1432}{4096}$ which is approximately equal to .34. Can you guess what the true area will approach? ***(Answer: using our successive approximations, the area will approach the value*** $1/3$. ***)***

We call this process of finding the area under a curve ***integration of*** $f(x)$ ***over the interval*** $[0,1]$.

Applications of Integral Calculus

We have not yet developed any computational machinery for computing ***derivatives*** and ***integrals*** so we will just state one popular application of integral calculus that relates the derivative and integrals of a function.

Example 2:

There are quite a few applications of calculus in business. One of these is the cost function $C(x)$ of producing x items of a product. It can be shown that the derivative of the cost function that gives the slope of the tangent line is another function that that gives the cost to produce an additional unit of the product. This is called the ***marginal cost*** and is a very important piece of information for management to have. Conversely, if one knows the marginal cost as a function of x, then finding the area under the curve of the function will give back the cost function $C(x)$.

Lesson Summary

1. We used linear approximations to study the limit process.
2. We computed approximations for the slope of tangent lines to a graph.
3. We analyzed applications of differential calculus.
4. We analyzed applications of integral calculus.

Review Questions

1. For the function $f(x) = x^2$ approximate the slope of the tangent line to the graph at the point $(3, 9)$.
   a. Use the following set of $x-$values to generate the sequence of secant line slopes:
   $$x = 2.9, 2.95, 2.975, 2.995, 2.999.$$
   b. What value does the sequence of slopes approach?
2. Consider the function $f(x) = x^2$.
   a. For what values of x would you expect the slope of the tangent line to be negative?
   b. For what value of x would you expect the tangent line to have slope $m = 0$?
   c. Give an example of a function that has two different horizontal tangent lines?
3. Consider the function $p(x) = x^3 - x$. Generate the graph of $p(x)$ using your calculator.
   a. Approximate the slope of the tangent line to the graph at the point $(2, 6)$. Use the following set of $x-$values to generate the sequence of secant line slopes. $x = 2.1, 2.05, 2.005, 2.001, 2.0001$.
   b. For what values of x do the tangent lines appear to have slope of 0? (Hint: Use the calculate function in your calculator to approximate the $x-$values.)
   c. For what values of x do the tangent lines appear to have positive slope?
   d. For what values of x do the tangent lines appear to have negative slope?
4. The cost of producing $xHi - Fi$ stereo receivers by Yamaha each week is modeled by the following function:
   $$C(x) = 850 + 200x - .3x^2.$$
   a. Generate the graph of $C(x)$ using your calculator. (Hint: Change your viewing window to reflect the high y values.)
   b. For what number of units will the function be maximized?
   c. Estimate the slope of the tangent line at $x = 200, 300, 400$.
   d. Where is marginal cost positive?
5. Find the area under the curve of $f(x) = x^2$ from $x = 1$ to $x = 3$. Use a rectangle method that uses the minimum value of the function within sub-intervals. Produce the approximation for each case of the subinterval cases.
   a. four sub-intervals.
   b. eight sub-intervals.
   c. Repeat part a. using a Mid-Point Value of the function within each sub-interval.
   d. Which of the answers in a. - c. provide the best estimate of the actual area?
6. Consider the function $p(x) = -x^3 + 4x$.
   a. Find the area under the curve from $x = 0$ to $x = 1$.

b. Can you find the area under the curve from $x = -1$ to $x = 0$. Why or why not? What is problematic for this computation?

7. Find the area under the curve of $f(x) = \sqrt{x}$ from $x = 1$ to $x = 4$. Use the Max Value rectangle method with six sub-intervals to compute the area.
8. The Eiffel Tower is 320 meters high. Suppose that you drop a ball off the top of the tower. The distance that it falls is a function of time and is given by $s(t) = 4.9t^2$.

Find the velocity of the ball after 4 seconds. (Hint: the average velocity for a time interval is **average velocity** = **change in distance/change in time**. Investigate the average velocity for t intervals close to $t = 4$ such as $3.9 \leq t \leq 4$ and closer and see if a pattern is evident.)

1.5 Finding Limits

Learning Objectives

A student will be able to:

- Find the limit of a function numerically.
- Find the limit of a function using a graph.
- Identify cases when limits do not exist.
- Use the formal definition of a limit to solve limit problems.

Introduction

In this lesson we will continue our discussion of the limiting process we introduced in Lesson 1.4. We will examine numerical and graphical techniques to find limits where they exist and also to examine examples where limits do not exist. We will conclude the lesson with a more precise definition of limits.

Let's start with the notation that we will use to denote limits. We indicate the limit of a function as the x values approach a particular value of x, say a, as

$$\lim_{x \to a} f(x).$$

So, in the example from Lesson 1.3 concerning the function $f(x) = x^2$, we took points that got closer to the point on the graph $(1,1)$ and observed the sequence of slope values of the corresponding secant lines. Using our limit notation here, we would write

$$\lim_{x \to 1} \frac{x^2-1}{x-1}.$$

Recall also that we found that the slope values tended to the value $x = 2$; hence using our notation we can write

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$$

Finding Limits Numerically

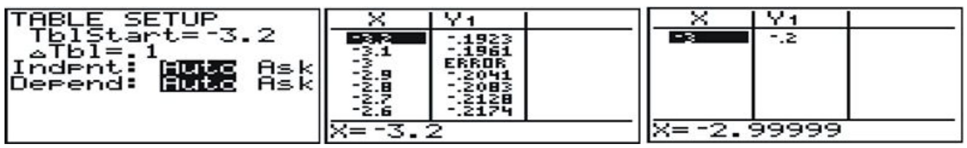
In our example in Lesson 1.3 we used this approach to find that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$. Let's apply this technique to a more complicated function.

Consider the rational function $f(x) = \frac{x+3}{x^2+x-6}$. Let's find the following limit:

$$\lim_{x \to -3} \frac{x+3}{x^2+x-6}.$$

Unlike our simple quadratic function, $f(x) = x^2$, it is tedious to compute the points manually. So let's use the **[TABLE]** function of our calculator. Enter the equation in your calculator and examine the table of points of the function. Do you notice anything unusual about the points? ***(Answer: There are error readings indicated for*** $x = -3, 2$ ***because the function is not defined at these values.)***

Even though the function is not defined at $x = -3$, we can still use the calculator to read the $y-$values for x values very close to $x = -3$. Press **2ND [TBLSET]** and set **Tblstart** to -3.2 and $\triangle$ to 0.1 (see screen on left below). The resulting table appears in the middle below.



Can you guess the value of $\lim_{x \to -3} \frac{x+3}{x^2+x-6}$? If you guessed $-.20 = -(1/5)$ you would be correct. Before we finalize our answer, let's get even closer to $x = -3$ and determine its function value using the **[CALC VALUE]** tool.

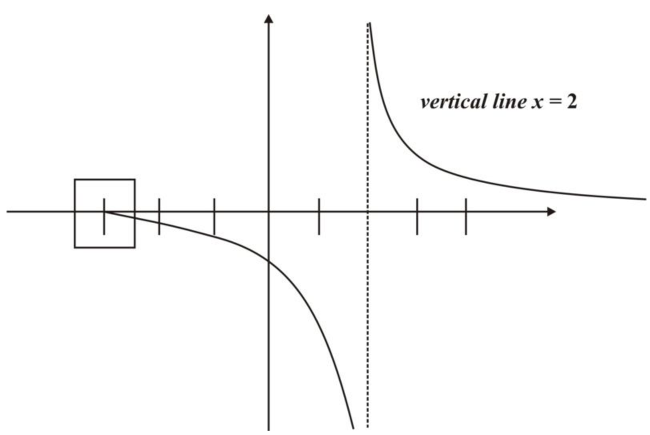
Press **2ND [TBLSET]** and change **Indpnt** from **Auto** to **Ask**. Now when you go to the table, enter $x = -2.99999$. and press **[ENTER]** and you will see the screen on the right above. Press **[ENTER]** and see that the function value is $x = -0.2$, which is the closest the calculator can display in the four decimal places allotted in the table. So our guess is correct and $\lim_{x \to -3} \frac{x+3}{x^2+x-6} = -\frac{1}{5}$.

Finding Limits Graphically

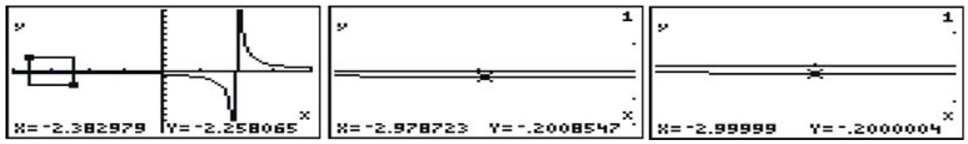
Let's continue with the same problem but now let's focus on using the graph of the function to determine its limit.

$$\lim_{x \to -3} \frac{x+3}{x^2+x-6}$$

We enter the function in the $Y =$ menu and sketch the graph. Since we are interested in the value of the function for x close to $x = -3$, we will look to **[ZOOM]** in on the graph at that point.



Our graph above is set to the normal viewing window $[-10, 10]$. Hence the values of the function appear to be very close to 0. But in our numerical example, we found that the function values approached $-.20 = -(1/5)$. To see this graphically, we can use the **[ZOOM]** and **[TRACE]** function of our calculator. Begin by choosing **[ZOOM]** function and choose **[BOX]**. Using the directional arrows to move the cursor, make a box around the x value -3. (See the screen on the left below Press **[ENTER]** and **[TRACE]** and you will see the screen in the middle below.) In **[TRACE]** mode, type the number -2.99999 and press **[ENTER]**. You will see a screen like the one on the right below.

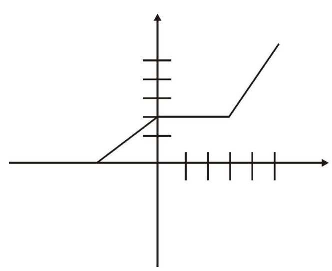


The graphing calculator will allow us to calculate limits graphically, provided that we have the function rule for the function so that we can enter its equation into the calculator. What if we have only a graph given to us and we are asked to find certain limits?

It turns out that we will need to have pretty accurate graphs that include sufficient detail about the location of data points. Consider the following example.

Example 1:

Find $\lim_{x\to 3} f(x)$ for the function pictured here. Assume units of value 1 for each unit on the axes.



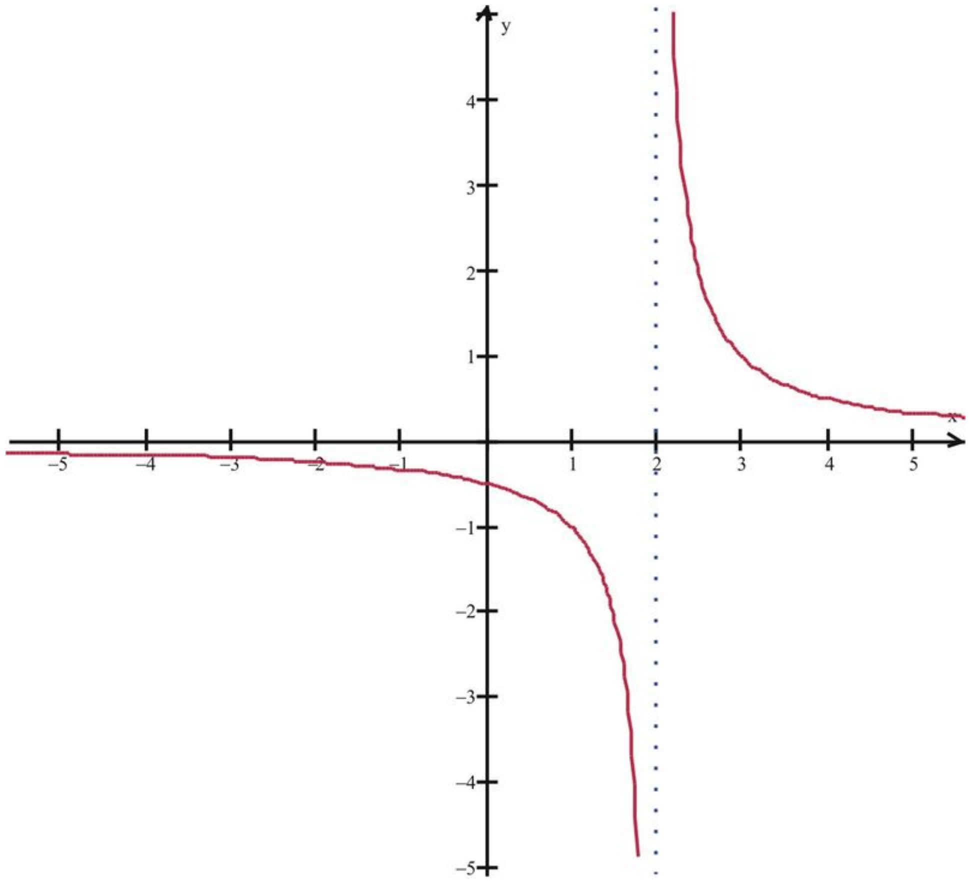
By inspection, we see that as we approach the value $x = 3$ from the left, we do so along what appears to be a portion of the horizontal line $y = 2$. We see that as we approach the value $x = 3$ from the right, we do so along a line segment having positive slope. In either case, the y values of $f(x)$ approaches $y = 2$.

Nonexistent Limits

We sometimes have functions where $\lim_{x\to a} f(x)$ does not exist. We have already seen an example of a function where our a value was not in the domain of the function. In particular, the function was not defined for $x = -3, 2$, but we could still find the limit as $x \to -3$.

$$\lim_{x\to -3} \frac{x+3}{x^2+x-6} = -\frac{1}{5}$$

What do you think the limit will be as we let $x \to 2$?



$$\lim_{x \to 2} \frac{x+3}{x^2+x-6}$$

Our inspection of the graph suggests that the function around $x = 2$ does not appear to approach a particular value. For $x > 2$, the points all lie in the first quadrant and appear to grow very quickly to large positive numbers as we get close to $x = 2$. Alternatively, for $x < 2$ we see that the points all lie in the fourth quadrant and decrease to large negative numbers. If we inspect actual values very close to $x = 2$ we can see that the values of the function do not approach a particular value.

x	y
1.999	-1000
1.9999	-10000
2	ERROR
2.001	1000
2.0001	10000

For this example, we say that $\lim_{x \to 2} \frac{x+3}{x^2+x-6}$ does not exist.

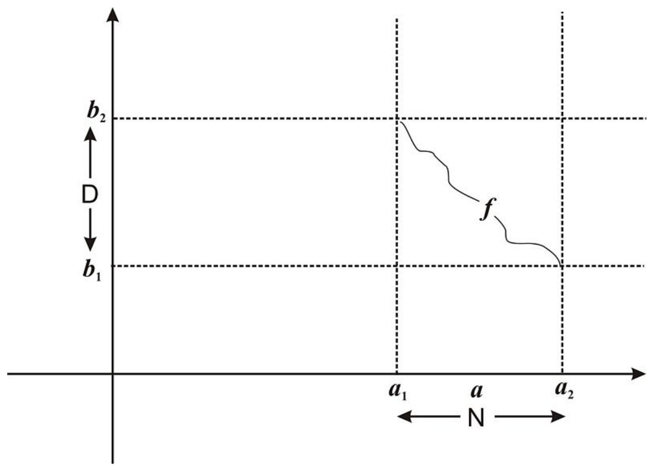
Formal Definition of a Limit

We conclude this lesson with a formal definition of a limit.

Definition:

We say that the limit of a function $f(x)$ at a is L, written as $\lim_{x\to a} f(x) = L$, if for every open interval D of L, there exists an open interval N of a, that does not include a, such that $f(x)$ is in D for every x in N.

This definition is somewhat intuitive to us given the examples we have covered. Geometrically, the definition means that for any lines $y = b_1, y = b_2$ below and above the line $y = L$, there exist vertical lines $x = a_1, x = a_2$ to the left and right of $x = a$ so that the graph of $f(x)$ between $x = a_1$ and $x = a_2$ lies between the lines $y = b_1$ and $y = b_2$. The key phrase in the above statement is "for every open interval D", which means that even if D is very, very small (that is, $f(x)$ is very, very close to L), it still is possible to find interval N where $f(x)$ is defined for all values except possibly $x = a$.



Example 2:

Use the definition of a limit to prove that

$$\lim_{x\to 3}(2x+1) = 7.$$

We need to show that for each open interval of 7, we can find an open neighborhood of 3, that does not include 3, so that all x in the open neighborhood map into the open interval of 7.

Equivalently, we must show that for every interval of 7, say $(7-\varepsilon, 7+\varepsilon)$, we can find an interval of 3, say $(3-\delta, 3+\delta)$, such that $(7-\varepsilon < 2x+1 < 7+\varepsilon)$ whenever $(3-\delta < x < 3+\delta)$.

The first inequality is equivalent to $6-\varepsilon < 2x < 6+\varepsilon$ and solving for x, we have

$$3-\frac{\varepsilon}{2} < x < 3+\frac{\varepsilon}{2}.$$

Hence if we take $\delta = \frac{\varepsilon}{2}$, we will have $3-\delta < x < 3+\delta \Rightarrow 7-\varepsilon < 2x+1 < 7+\varepsilon$.

Fortunately, we do not have to do this to evaluate limits. In Lesson 1.6 we will learn several rules that will make the task manageable.

Lesson Summary

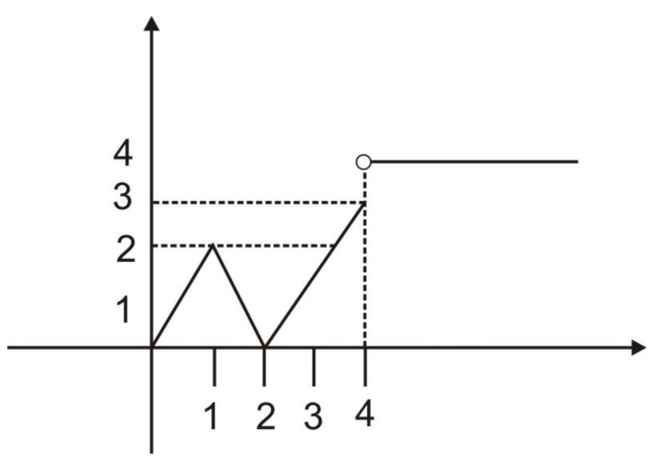
1. We learned to find the limit of a function numerically.
2. We learned to find the limit of a function using a graph.
3. We identified cases when limits do not exist.
4. We used the formal definition of a limit to solve limit problems.

Multimedia Links

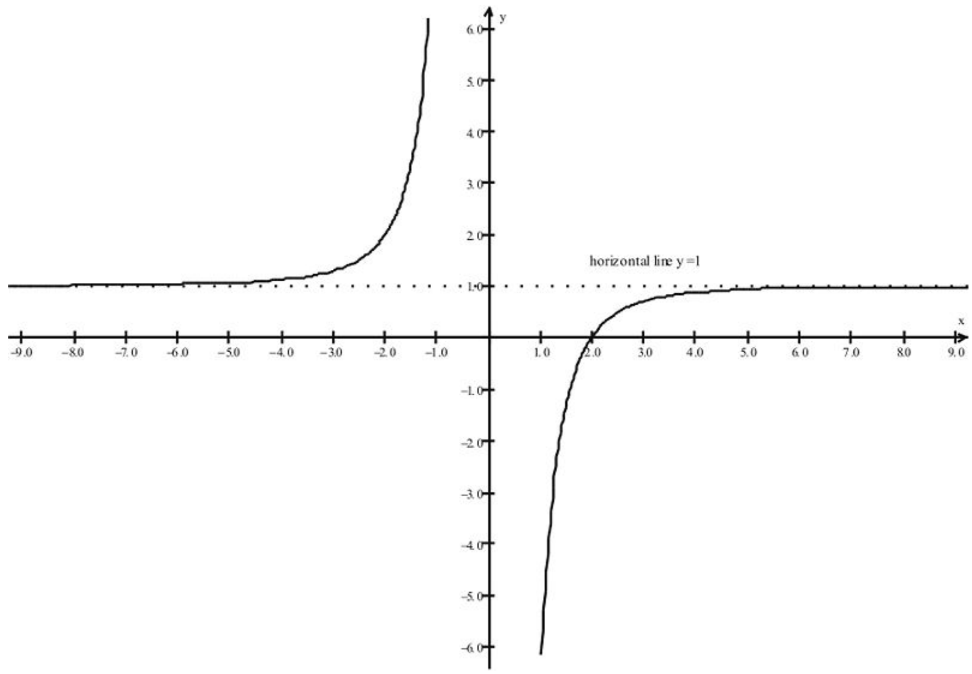
For another look at the definition of a limit, the series of videos at Tutorials for the Calculus Phobe has a nice intuitive introduction to this fundamental concept (despite the whimsical name). If you want to experiment with limits yourself, follow the sequence of activities using a graphing applet at Informal Limits . Directions for using the graphing applets at this very useful site are also available at Applet Intro .

Review Questions

1. Use a table of values to find $\lim_{x \to -2} \frac{x^2-4}{x+2}$.
   a. Use $x-$values of $x = -1.9, -1.99, -1.999, -2.1, -2.099, -2.0099$.
   b. What value does the sequence of values approach?
2. Use a table of values to find $\lim_{x \to \frac{1}{2}} \frac{2x-1}{2x^2+3x-2}$.
   a. Use $x-$values of $x = .49, .495, .49999, .51, .5099, .500001$.
   b. What value does the sequence of values approach?
3. Consider the function $p(x) = 3x^3 - 3x$. Generate the graph of $p(x)$ using your calculator. Find each of the following limits if they exist. Use tables with appropriate x values to determine the limits.
   a. $\lim_{x \to 4}(3x^3 - 3x)$
   b. $\lim_{x \to -4}(3x^3 - 3x)$
   c. $\lim_{x \to 0}(3x^3 - 3x)$
   d. Find the values of the function corresponding to $x = 4, -4, 0$. How do these function values compare to the limits you found in #a-c? Explain your answer.
4. Examine the graph of $f(x)$ below to approximate each of the following limits if they exist.



   a. $\lim_{x \to 3} f(x)$
   b. $\lim_{x \to 2} f(x)$
   c. $\lim_{x \to 1} f(x)$
   d. $\lim_{x \to 4} f(x)$
5. Examine the graph of $f(x)$ below to approximate each of the following limits if they exist.



a. $\lim_{x\to 2} f(x)$
b. $\lim_{x\to 0} f(x)$
c. $\lim_{x\to 4} f(x)$
d. $\lim_{x\to 50} f(x)$

In problems #6-8, determine if the indicated limit exists. Provide a numerical argument to justify your answer.

6. $\lim_{x\to 2}(x^2+3)$
7. $\lim_{x\to -1} \frac{x+1}{x^2-1}$
8. $\lim_{x\to 2} \sqrt{-2x+5}$

In problems #9-10, determine if the indicated limit exists. Provide a graphical argument to justify your answer. (Hint: Make use of the **[ZOOM]** and **[TABLE]** functions of your calculator to view functions values close to the indicated *x* value.

9. $\lim_{x\to 4}(x^2+3x)$
10. $\lim_{x\to -1} \frac{|x+1|}{x+1}$

1.6 Evaluating Limits

Learning Objectives

A student will be able to:

- Find the limit of basic functions.
- Use properties of limits to find limits of polynomial, rational and radical functions.
- Find limits of composite functions.
- Find limits of trigonometric functions.
- Use the Squeeze Theorem to find limits.

Introduction

In this lesson we will continue our discussion of limits and focus on ways to evaluate limits. We will observe the limits of a few basic functions and then introduce a set of laws for working with limits. We will conclude the lesson with a theorem that will allow us to use an indirect method to find the limit of a function.

Direct Substitution and Basic Limits

Let's begin with some observations about limits of basic functions. Consider the following limit problems:

$$\lim_{x \to 2} 5,$$
$$\lim_{x \to 4} x.$$

These are examples of limits of basic constant and linear functions, $f(x) = c$ and $f(x) = mx + b$.

We note that each of these functions are defined for all real numbers. If we apply our techniques for finding the limits we see that

$$\lim_{x \to 2} 5 = 5,$$
$$\lim_{x \to 4} x = 4,$$

and observe that for each the limit equals the value of the function at the $x-$value of interest:

$$\lim_{x \to 2} 5 = f(5) = 5,$$
$$\lim_{x \to 4} x = f(4) = 4.$$

Hence $\lim_{x \to a} f(x) = f(a)$. This will also be true for some of our other basic functions, in particular all polynomial and radical functions, provided that the function is defined at $x = a$. For example, $\lim_{x \to 3} x^3 = f(3) = 27$ and $\lim_{x \to 4} \sqrt{x} = f(4) = 2$. The properties of functions that make these facts true will be discussed in Lesson 1.7. For now, we wish to use this idea for evaluating limits of basic functions. However, in order to evaluate limits of more complex function we will need some properties of limits, just as we needed laws for dealing with complex problems involving exponents. A simple example illustrates the need we have for such laws.

Example 1:

Evaluate $\lim_{x \to 2}(x^3 + \sqrt{2x})$. The problem here is that while we know that the limit of each individual function of the sum exists, $\lim_{x \to 2} x^3 = 8$ and $\lim_{x \to 2} \sqrt{2x} = 2$, our basic limits above do not tell us what happens when we find the limit of a sum of functions. We will state a set of properties for dealing with such sophisticated functions.

Properties of Limits

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then

1. $\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$ where c is a real number,
2. $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is a real number,
3. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,
4. $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$,
5. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided that $\lim_{x \to a} g(x) \neq 0$.

With these properties we can evaluate a wide range of polynomial and radical functions. Recalling our example above, we see that

$$\lim_{x \to 2}(x^3 + \sqrt{2x}) = \lim_{x \to 2}(x^3) + \lim_{x \to 2}(\sqrt{2x}) = 8 + 2 = 10.$$

Find the following limit if it exists:

$$\lim_{x \to -4}(2x^2 - \sqrt{-x}).$$

Since the limit of each function within the parentheses exists, we can apply our properties and find

$$\lim_{x \to -4}(2x^2 - \sqrt{-x}) = \lim_{x \to -4} 2x^2 - \lim_{x \to -4} \sqrt{-x}.$$

Observe that the second limit, $\lim_{x \to -4} \sqrt{-x}$, is an application of Law #2 with $n = \frac{1}{2}$. So we have $\lim_{x \to -4}(2x^2 - \sqrt{-x}) = \lim_{x \to -4} 2x^2 - \lim_{x \to -4} \sqrt{-x} = 32 - 2 = 30$.

In most cases of sophisticated functions, we simplify the task by applying the Properties as indicated. We want to examine a few exceptions to these rules that will require additional analysis.

Strategies for Evaluating Limits of Rational Functions

Let's recall our example

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

We saw that the function did not have to be defined at a particular value for the limit to exist. In this example, the function was not defined for $x = 1$. However we were able to evaluate the limit numerically by checking functional values around $x = 1$ and found $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Note that if we tried to evaluate by direct substitution, we would get the quantity $0/0$, which we refer to as an ***indeterminate form.*** In particular, Property #5 for finding limits does not apply since $\lim_{x \to 1}(x-1) = 0$. Hence in order to evaluate the limit without using numerical or graphical techniques we make the following observation. The numerator of the function can be factored, with one factor common to the denominator, and the fraction simplified as follows:

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

In making this simplification, we are indicating that the original function can be viewed as a linear function for x values close to but not equal to 1, that is,

$\frac{x^2-1}{x-1} = x+1$ for $x \neq 1$. In terms of our limits, we can say

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1}(x+1) = 1+1 = 2.$$

Example 2:

Find $\lim_{x \to 0} \frac{x^2+5x}{x}$.

This is another case where direct substitution to evaluate the limit gives the indeterminate form $0/0$. Reducing the fraction as before gives:

$$\lim_{x \to 0} \frac{x^2+5x}{x} = \lim_{x \to 0}(x+5) = 5.$$

Example 3:

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}.$$

In order to evaluate the limit, we need to recall that the difference of squares of real numbers can be factored as $x^2 - y^2 = (x+y)(x-y)$.

We then rewrite and simplify the original function as follows:

$$\frac{\sqrt{x}-3}{x-9} = \frac{\sqrt{x}-3}{(\sqrt{x}+3)(\sqrt{x}-3)} = \frac{1}{(\sqrt{x}+3)}.$$

Hence $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$.

You will solve similar examples in the homework where some clever applications of factoring to reduce fractions will enable you to evaluate the limit.

Limits of Composite Functions

While we can use the Properties to find limits of composite functions, composite functions will present some difficulties that we will fully discuss in the next Lesson. We can illustrate with the following examples, one where the limit exists and the other where the limit does not exist.

Example 4:

Consider $f(x) = \frac{1}{x+1}, g(x) = x^2$. Find $\lim_{x \to -1}(f \circ g)(x)$.

We see that $(f \circ g)(x) = \frac{1}{x^2+1}$ and note that property #5 does hold. Hence by direct substitution we have $\lim_{x \to -1}(f \circ g)(x) = \frac{1}{(-1)^2+1} = \frac{1}{2}$.

Example 5:

Consider $f(x) = \frac{1}{x+1}, g(x) = -1$. Then we have that $f(g(x))$ is undefined and we get the indeterminate form $1/0$. Hence $\lim_{x \to -1}(f \circ g)(x)$ does not exist.

Limits of Trigonometric Functions

In evaluating limits of trigonometric functions we will look to rely more on numerical and graphical techniques due to the unique behavior of these functions. Let's look at a couple of examples.

Example 6:

Find $\lim_{x \to 0} \sin(x)$.

We can find this limit by observing the graph of the sine function and using the **[CALC VALUE]** function of our calculator to show that $\lim_{x \to 0} \sin x = 0$.

While we could have found the limit by direct substitution, in general, when dealing with trigonometric functions, we will rely less on formal properties of limits for finding limits of trigonometric functions and more on our graphing and numerical techniques.

The following theorem provides us a way to evaluate limits of complex trigonometric expressions.

Squeeze Theorem

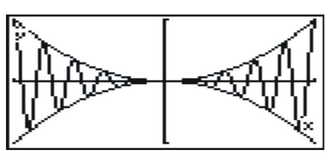
Suppose that $f(x) \le g(x) \le h(x)$ for x near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

Then $\lim_{x \to a} g(x) = L$.

In other words, if we can find bounds for a function that have the same limit, then the limit of the function that they bound must have the same limit.

Example 7:

Find $\lim_{x \to 0} x^2 \cos(10\pi x)$.



From the graph we note that:

1. The function is bounded by the graphs of x^2 and $-x^2$
2. $\lim_{x \to 0} x^2 = \lim_{x \to 0}(-x^2) = 0$.

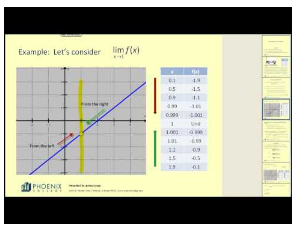
Hence the Squeeze Theorem applies and we conclude that $\lim_{x \to 0} x^2 \cos(10\pi x) = 0$.

Lesson Summary

1. We learned to find the limit of basic functions.
2. We learned to find the limit of polynomial, rational and radical functions.
3. We learned how to find limits of composite and trigonometric functions.
4. We used the Squeeze Theorem to find special limits.

Multimedia Links

For an introduction to finding limits **(1.0)**, see Math Video Tutorials by James Sousa, Introduction to Limits (8:46).



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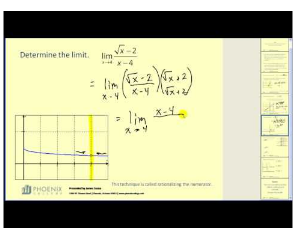
For more examples finding the limits of functions **(1.1)(1.2)**, see Math Video Tutorials by James Sousa, Determining the Limits of Functions (8:34).



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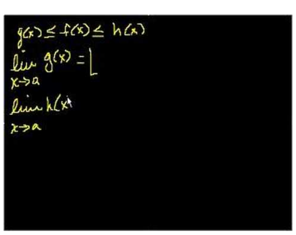
For a brief, intuitive introduction to the Squeeze Theorem using everyday examples **(1.1)**, see Khan Academy Squ eeze Theorem (7:37).



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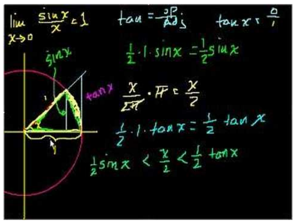
This video includes a graphical presentation of the Squeeze Theorem. This video serves as an introduction to another (much longer) video, Khan Academy Proof of Limit x -}textgreater 0 sin(x)/x (18:05).



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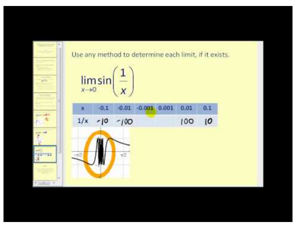
For a video on determining limits of trigonometric functions **(1.2)(1.3)**, see Math Video Tutorials by James Sousa, Determining Limits that Involve Trigonometric Functions (7:20).



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Review Questions

Find each of the following limits if they exist.

1. $\lim_{x\to 2}(x^2-3x+4)$
2. $\lim_{x\to 4}\frac{x^2-16}{x-4}$
3. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$
4. $\lim_{x\to -1}\frac{x-2}{x+1}$
5. $\lim_{x\to -1}\frac{10x-2}{3x+1}$
6. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$
7. $\lim_{x\to 5}\frac{x^2-25}{x^3-125}$
8. Consider function $f(x)$ such that $5x-11 \le f(x) \le x^2-4x+9$ for $x \ge 0$. Use the Squeeze Theorem to find $\lim_{x\to 5} f(x)$.
9. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^4 \sin(\frac{1}{x}) = 0$

1.7 Continuity

Learning Objectives

A student will be able to:

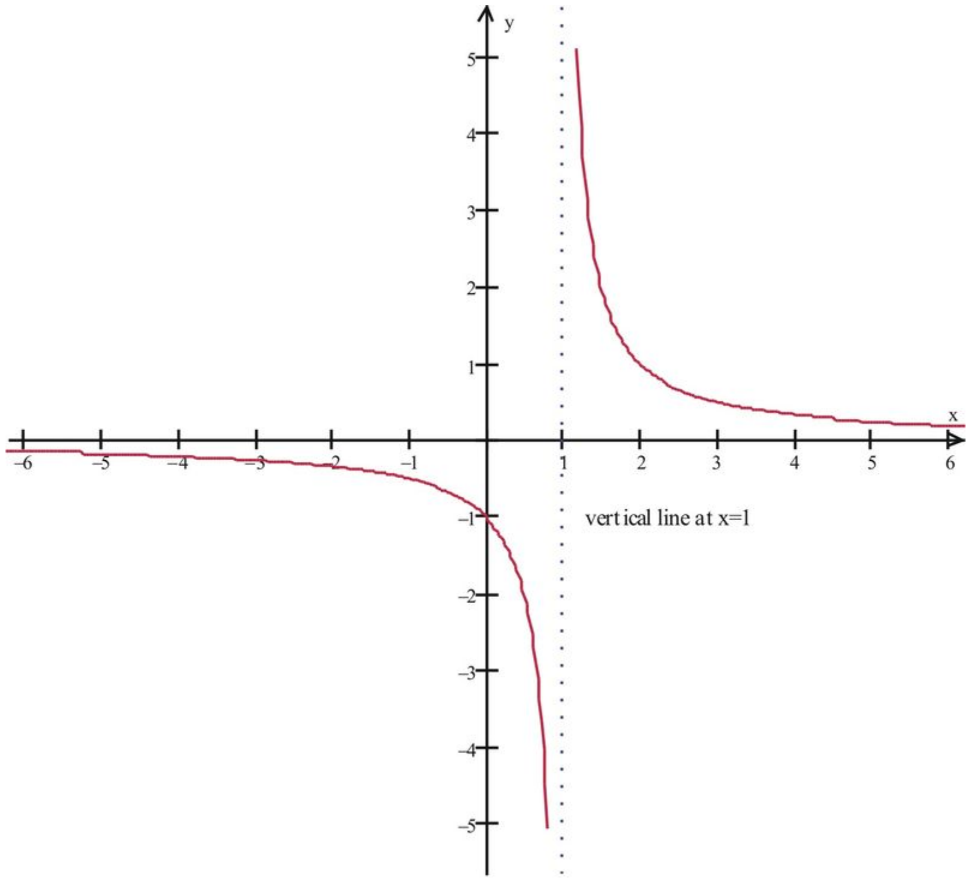
- Learn to examine continuity of functions.
- Find one-sided limits.
- Understand properties of continuous functions.
- Solve problems using the Min-Max theorem.
- Solve problems using the Intermediate Value Theorem.

Introduction

In this lesson we will discuss the property of continuity of functions and examine some very important implications. Let's start with an example of a rational function and observe its graph. Consider the following function:

$$f(x) = (x+1)/(x^2-1).$$

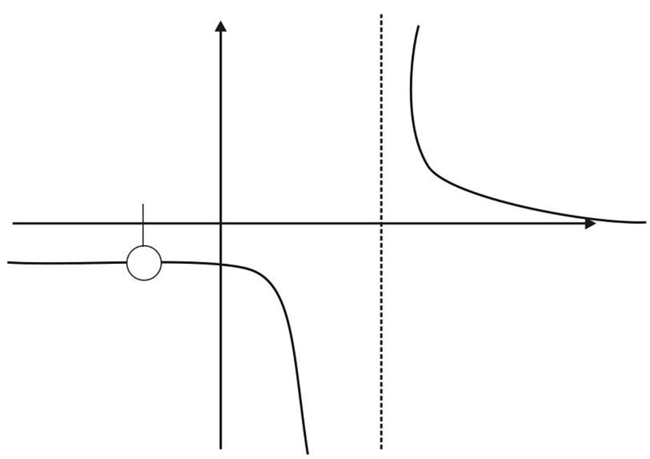
We know from our study of domains that in order for the function to be defined, we must use $x \neq -1, 1$. Yet when we generate the graph of the function (using the standard viewing window), we get the following picture that appears to be defined at $x = -1$:



The seeming contradiction is due to the fact that our original function had a common factor in the numerator and denominator, $x+1$, that cancelled out and gave us a picture that appears to be the graph of $f(x) = 1/(x-1)$.

But what we actually have is the original function, $f(x) = (x+1)/(x^2-1)$, that we know is not defined at $x = -1$. At $x = -1$, we have a hole in the graph, or a discontinuity of the function at $x = -1$. That is, the function is defined for all other $x-$values close to $x = -1$.

Loosely speaking, if we were to hand-draw the graph, we would need to take our pencil off the page when we got to this hole, leaving a gap in the graph as indicated:



Now we will formalize the property of continuity of a function and provide a test for determining when we have continuous functions.

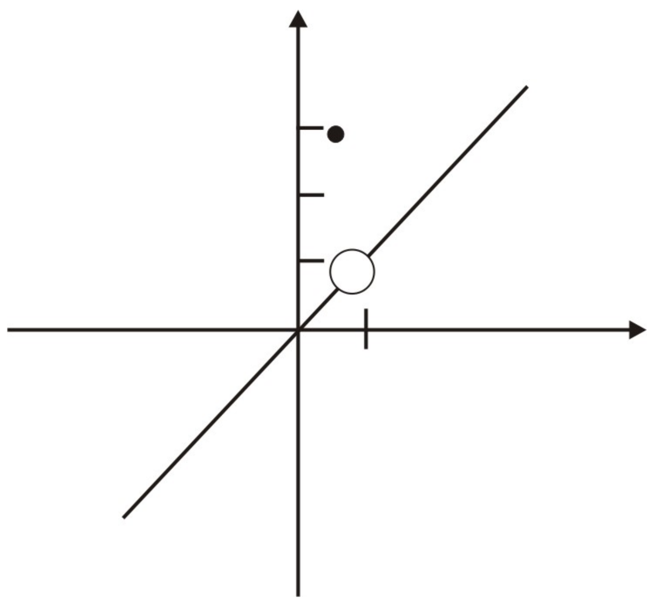
Continuity of a Function

Definition:

The function $f(x)$ is ***continuous at*** $x = a$ if the following conditions all hold:

1. a is in the domain of $f(x)$;
2. $\lim_{x\to a} f(x)$ exists;
3. $\lim_{x\to a} f(x) = f(a)$

Note that it is possible to have functions where two of these conditions are satisfied but the third is not. Consider the piecewise function

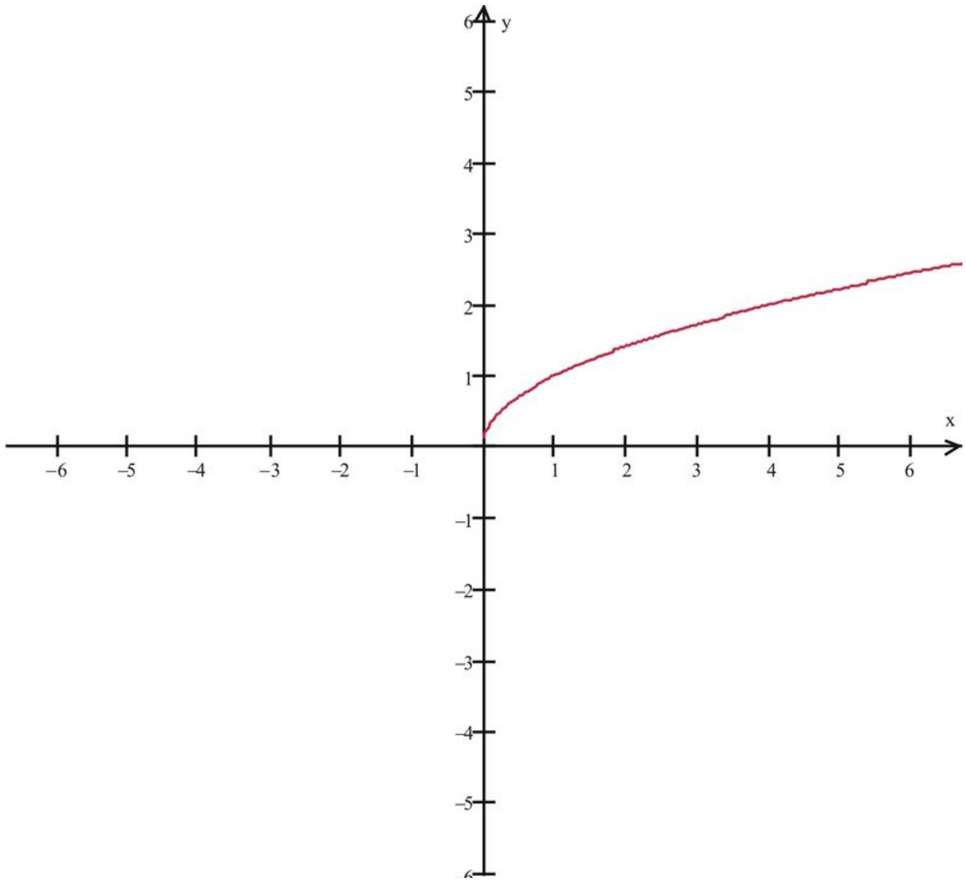


$$f(x) = \begin{cases} x, \text{if } x \neq 1 \\ 3, \text{if } x = 1 \end{cases}$$

In this example we have $\lim_{x\to 1} f(x)$ exists, $x = 1$ is in the domain of $f(x)$, but $\lim_{x\to 1} f(x) \neq f(1)$.

One-Sided Limits and Closed Intervals

Let's recall our basic square root function, $f(x) = \sqrt{x}$.



Since the domain of $f(x) = \sqrt{x}$ is $x \geq 0$, we see that that $\lim_{x \to 0} \sqrt{x}$ does not exist. Specifically, we cannot find open intervals around $x = 0$ that satisfy the limit definition. However we do note that as we approach $x = 0$ from the right-hand side, we see the successive values tending towards $x = 0$. This example provides some rationale for how we can define ***one-sided limits***.

Definition:

We say that the ***right-hand*** *limit* of a function $f(x)$ at a is b, written as $\lim_{x \to a^+} f(x) = b$, if for every open interval N of b, there exists an open interval $(a, a+\delta)$ contained in the domain of $f(x)$, such that $f(x)$ is in N for every x in $(a, a+\delta)$.

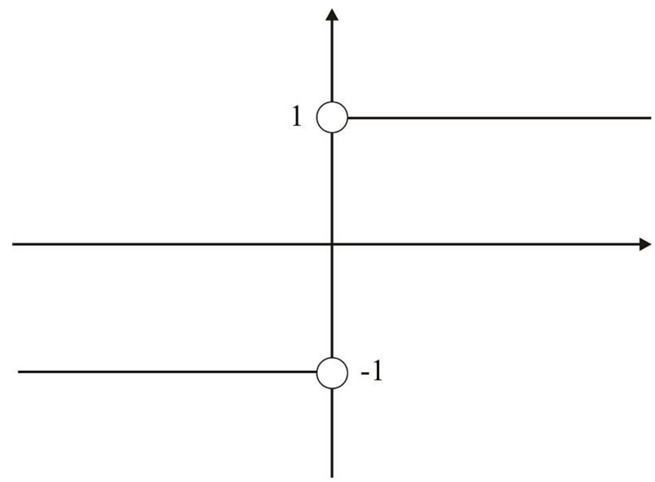
For the example above, we write $\lim_{x \to 0^+} \sqrt{x} = 0$.

Similarly, we say that the ***left-hand limit*** of $f(x)$ at a is b, written as $\lim_{x \to a^-} f(x) = b$, if for every open interval N of b there exists an open interval $(a-\delta, a)$ contained in the domain of $f(x)$, such that $f(x)$ is in N for every x in $(a-\delta, a)$.

Example 1:

Find $\lim_{x \to 0^+} \frac{x}{|x|}$.

The graph has a discontinuity at $x = 0$ as indicated:



We see that $\lim_{x\to 0^+} \frac{x}{|x|} = 1$ and also that $\lim_{x\to 0^-} \frac{x}{|x|} = -1$.

Properties of Continuous Functions

Let's recall our example of the limit of composite functions:

$$f(x) = 1/(x+1),\ g(x) = -1.$$

We saw that $f(g(x))$ is undefined and has the indeterminate form of $1/0$. Hence $\lim_{x\to -1}(f\circ g)(x)$ does not exist.

In general, we will require that f be continuous at $x = g(a)$ and $x = g(a)$ must be in the domain of $(f\circ g)$ in order for $\lim_{x\to a}(f\circ g)(x)$ to exist.

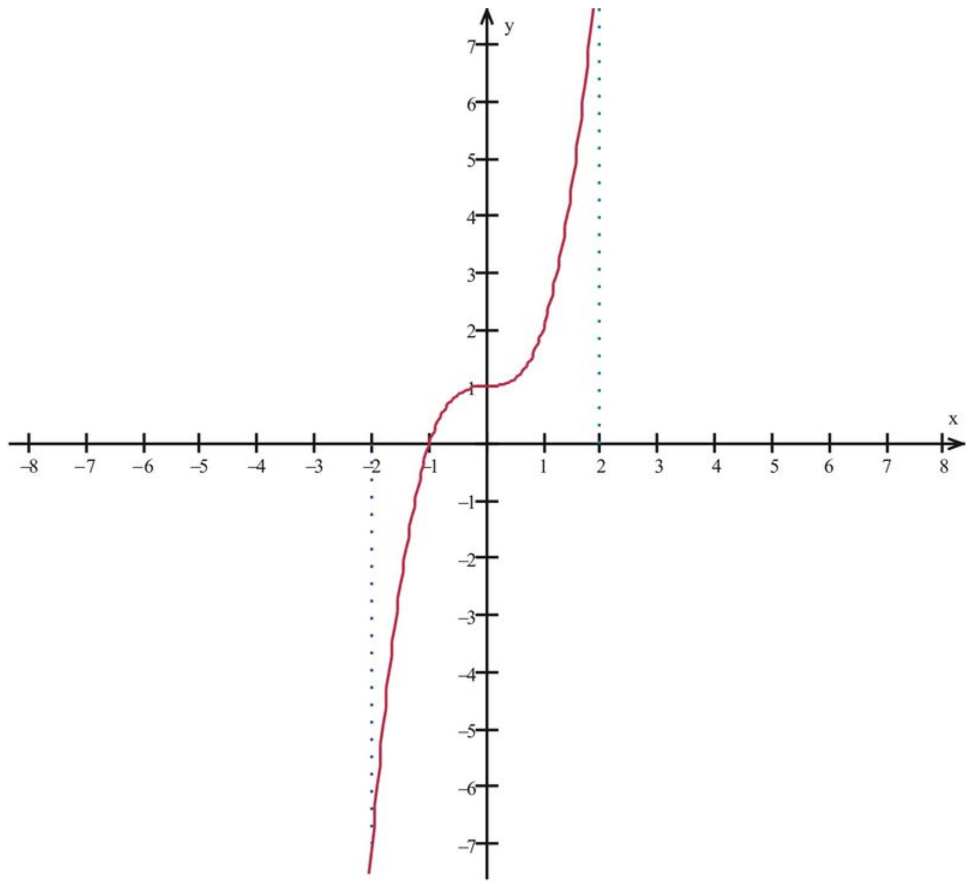
We will state the following theorem and delay its proof until Chapter 3 when we have learned more about real numbers.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I, then $f(x)$ has both a maximum value and a minimum value in I.

Example 2:

Consider $f(x) = x^3 + 1$ and interval $I = [-2, 2]$.

The function has a minimum value at value at $x = -2, f(-2) = -7$, and a maximum value at $x = 2$, where $f(2) = 9$



We will conclude this lesson with a theorem that will enable us to solve many practical problems such as finding zeros of functions and roots of equations.

Intermediate Value Theorem

If a function is continuous on a closed interval $[a,b]$, then the function assumes every value between $f(a)$ and $f(b)$.

The proof is left as an exercise with some hints provided. (Homework #10).

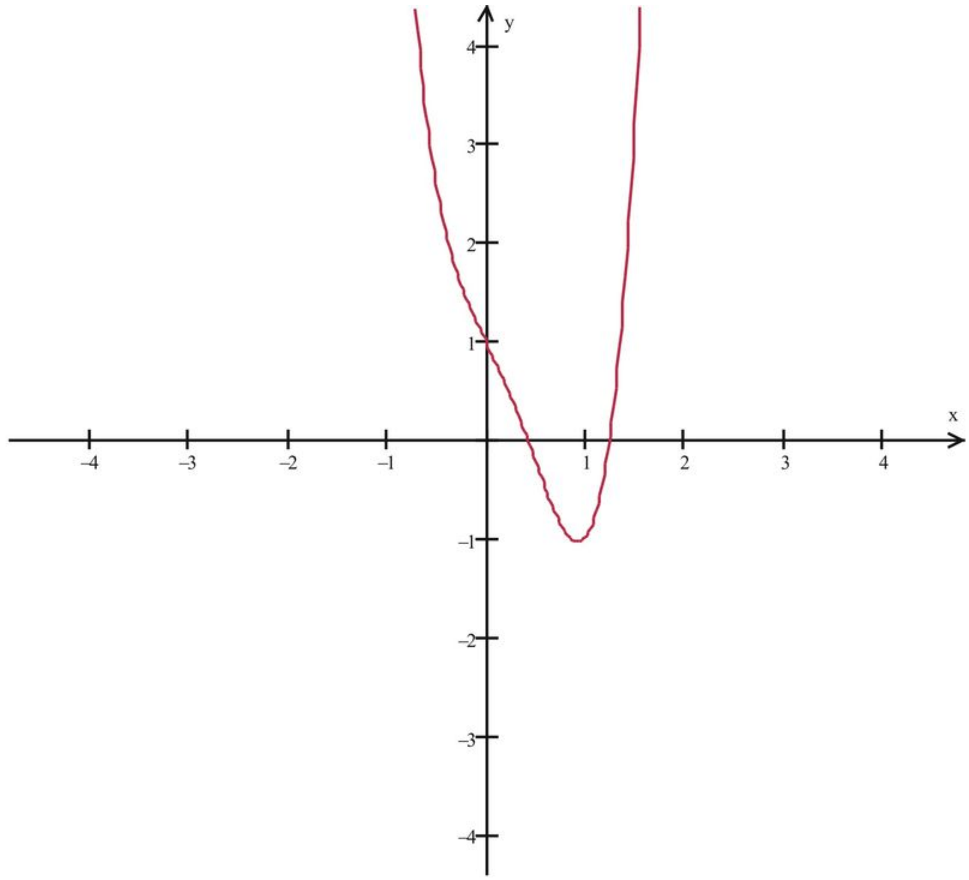
We can use the Intermediate Value Theorem to analyze and approximate zeros of functions.

Example 3:

Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.

$$f(x) = 3x^4 - 3x^3 - 2x + 1,\ (1,2)$$

We recall that the graph of this function is shaped somewhat like a parabola; viewing the graph in the standard window, we get the following graph:



Of course we could zoom in on the graph to see that the lowest point on the graph lies within the fourth quadrant, but let's use the **[CALC VALUE]** function of the calculator to verify that there is a zero in the interval $(1,2)$. In order to apply the Intermediate Value Theorem, we need to find a pair of $x-$values that have function values with different signs. Let's try some in the table below.

x	$f(x)$
1.1	$-.80$
1.2	$-.36$
1.3	.37

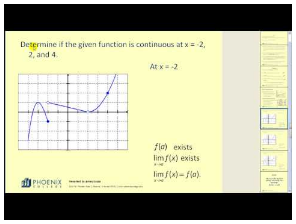
We see that the sign of the function values changes from negative to positive somewhere between 1.2 and 1.3. Hence, by the Intermediate Value theorem, there is some value c in the interval $(1.2,1.3)$ such that $f(c)=0$.

Lesson Summary

1. We learned to examine continuity of functions.
2. We learned to find one-sided limits.
3. We observed properties of continuous functions.
4. We solved problems using the Min-Max theorem.
5. We solved problems using the Intermediate Value Theorem.

Multimedia Links

For a presentation of continuity using limits **(2.0)**, see Math Video Tutorials by James Sousa, Continuity Using Limits (5:44).



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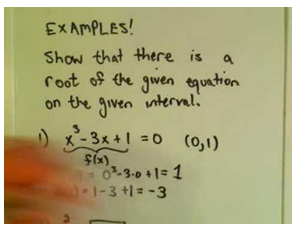
For a video presentation of the Intermediate Value Theorem **(3.0)**, see Just Math Tutoring, Intermediate Value Theorem (7:53).



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Review Questions

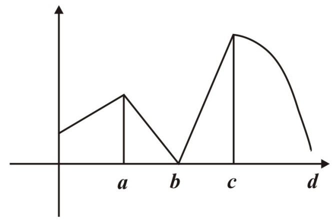
1. Generate the graph of $f(x)=(|x+1|)/(x+1)$ using your calculator and discuss the continuity of the function.
2. Generate the graph of $f(x)=(3x-6)/(x^2-4)$ using your calculator and discuss the continuity of the function.

Compute the limits in #3 - 6.

3. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}-1}}$
4. $\lim_{x\to 2^-} \frac{x^3-8}{|x-2|(x-2)}$
5. $\lim_{x\to 1^+} \frac{2x|x-1|}{x-1}$
6. $\lim_{x\to -2^-} \frac{|x+2|+x+2}{|x+2|-x-2}$

In problems 7 and 8, explain how you know that the function has a root in the given interval. (Hint: Use the Intermediate Value Function to show that there is at least one zero of the function in the indicated interval.):

7. $f(x)=x^3+2x^2-x+1$, in the interval $(-2,-3)$
8. $f(x)=\sqrt{x}-\sqrt[3]{x}-1$, in the interval $(9,10)$
9. State whether the indicated $x-$values correspond to maximum or minimum values of the function depicted below.



10. Prove the Intermediate Value Theorem: If a function is continuous on a closed interval $[a,b]$, then the function assumes every value between $f(a)$ and $f(b)$.

1.8 Infinite Limits

Learning Objectives

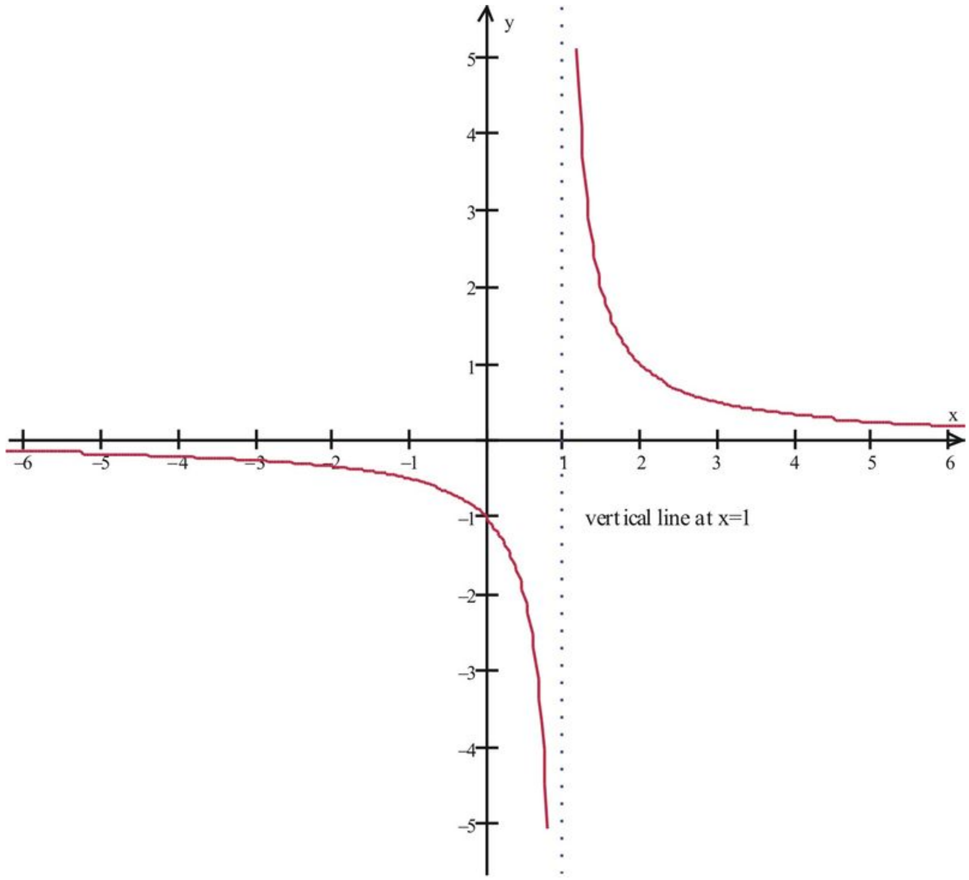
A student will be able to:

- Find infinite limits of functions.
- Analyze properties of infinite limits.
- Identify asymptotes of functions.
- Analyze end behavior of functions.

Introduction

In this lesson we will discuss infinite limits. In our discussion the notion of infinity is discussed in two contexts. First, we can discuss infinite limits in terms of the value a function as we increase x without bound. In this case we speak of the ***limit of*** $f(x)$ ***as*** x ***approaches*** ∞ and write $\lim_{x\to\infty} f(x)$. We could similarly refer to the ***limit of*** $f(x)$ ***as*** x ***approaches*** -∞ and write $\lim_{x\to-\infty} f(x)$.

The second context in which we speak of infinite limits involves situations where the function values increase without bound. For example, in the case of a rational function such as $f(x) = (x+1)/(x^2+1)$, a function we discussed in previous lessons:



At $x = 1$, we have the situation where the graph grows without bound in both a positive and a negative direction. We say that we have a vertical asymptote at $x = 1$, and this is indicated by the dotted line in the graph above.

In this example we note that $\lim_{x\to 1} f(x)$ does not exist. But we could compute both one-sided limits as follows.

$\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^-} f(x) = +\infty$.

More formally, we define these as follows:

Definition:

The right-hand limit of the function $f(x)$ at $x = a$ is infinite, and we write $\lim_{x\to a^+} f(x) = \infty$, if for every positive number k, there exists an open interval $(a, a+\delta)$ contained in the domain of $f(x)$, such that $f(x)$ is in (k, ∞) for every x in $(a, a+\delta)$.

The definition for negative infinite limits is similar.

Suppose we look at the function $f(x) = (x+1)/(x^2-1)$ and determine the infinite limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

We observe that as x increases in the positive direction, the function values tend to get smaller. The same is true if we decrease x in the negative direction. Some of these extreme values are indicated in the following table.

x	$f(x)$
100	.0101
200	.0053
-100	$-.0099$
-200	$-.005$

We observe that the values are getting closer to $f(x) = 0$. Hence $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Since our original function was roughly of the form $f(x) = \frac{1}{x}$, this enables us to determine limits for all other functions of the form $f(x) = \frac{1}{x^p}$ with $p > 0$. Specifically, we are able to conclude that $\lim_{x\to\infty} \frac{1}{x^p} = 0$. This shows how we can find infinite limits of functions by examining the ***end behavior*** of the function $f(x) = \frac{1}{x^p}$, $p > 0$.

The following example shows how we can use this fact in evaluating limits of rational functions.

Example 1:

Find $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1}$.

Solution:

Note that we have the indeterminate form, so Limit Property #5 does not hold. However, if we first divide both numerator and denominator by the quantity x^6, we will then have a function of the form

$$\frac{f(x)}{g(x)} = \frac{\frac{2x^3}{x^6} - \frac{x^2}{x^6} + \frac{x}{x^6} - \frac{1}{x^6}}{\frac{x^6}{x^6} - \frac{x^5}{x^6} + \frac{3x^4}{x^6} - \frac{2x}{x^6} + \frac{1}{x^6}} = \frac{\frac{2}{x^3} - \frac{1}{x^4} + \frac{1}{x^5} - \frac{1}{x^6}}{1 - \frac{1}{x} + \frac{3}{x^2} - \frac{2}{x^5} + \frac{1}{x^6}}.$$

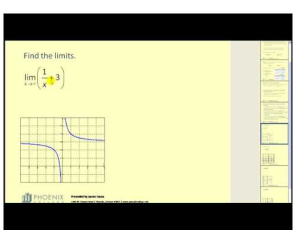
We observe that the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist. In particular, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$. Hence Property #5 now applies and we have $\lim_{x\to\infty} \frac{2x^3-x^2+x-1}{x^6-x^5+3x^4-2x+1} = \frac{0}{1} = 0$.

Lesson Summary

1. We learned to find infinite limits of functions.
2. We analyzed properties of infinite limits.
3. We identified asymptotes of functions.
4. We analyzed end behavior of functions.

Multimedia Links

For more examples of limits at infinity **(1.0)**, see Math Video Tutorials by James Sousa, Limits at Infinity (9:42).



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Review Questions

In problems 1 - 7, find the limits if they exist.

1. $\lim_{x\to 3^+} \frac{(x+2)^2}{(x-2)^2-1}$

2. $\lim_{x\to\infty} \frac{(x+2)^2}{(x-2)^2-1}$
3. $\lim_{x\to1^+} \frac{(x+2)^2}{(x-2)^2-1}$
4. $\lim_{x\to\infty} \frac{2x-1}{x+1}$
5. $\lim_{x\to-\infty} \frac{x^5+3x^4+1}{x^3-1}$
6. $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3}$
7. $\lim_{x\to\infty} \frac{2x^2-x+3}{x^5-2x^3+2x-3}$

In problems 8 - 10, analyze the given function and identify all asymptotes and the end behavior of the graph.

8. $f(x) = \frac{(x+4)^2}{(x-4)^2-1}$
9. $f(x) = -3x^3 - x^2 + 2x + 2$
10. $f(x) = \frac{2x^2-8}{x+2}$
11. Consider $f(x) = \frac{1}{x+1}, g(x) = x^2$. We previously found $\lim_{x\to-1}(f\circ g)(x) = \frac{1}{2}$. Find $\lim_{x\to-1}(g\circ f)(x)$.

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9726 .

CHAPTER 2 Derivatives

Chapter Outline

2.1 Tangent Lines and Rates of Change

Learning Objectives

A student will be able to:

- Demonstrate an understanding of the slope of the tangent line to the graph.
- Demonstrate an understanding of the instantaneous rate of change.

A car speeding down the street, the inflation of currency, the number of bacteria in a culture, and the AC voltage of an electric signal are all examples of quantities that change with time. In this section, we will study the rate of change of a quantity and how is it related to the tangent lines on a curve.

The Tangent Line

If two points $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two different points of the curve $y = f(x)$, then the slope of the secant line connecting the two points is given by

$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad (1)$$

Now if we let the the point x_1 approach x_0, Q will approach P along the graph f. Thus the slope of the secant line will gradually approach the slope of the tangent line as x_1 approaches x_0. Therefore (1) becomes

$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (2)$$

If we let $h = x_1 - x_0$, then $x_1 = x_0 + h$ and $h \to 0$ becomes equivalent to $x_1 \to x_0$, so (2) becomes

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If the point $P(x_0, y_0)$ is on the curve f, then the tangent line at P has a slope that is given by

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists.

Recall from algebra that the *point-slope* form for the tangent line is given by

$$y-y_0 = m_{tan}(x-x_0).$$

Example 1:

Find the slope of the tangent line to the curve $f(x)=x^3$ passing through point $P(2,8)$.

Solution:

Since $P(x_0,y_0)=(2,8)$, using the slope of the tangent equation,

$$m_{tan} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

we get

$$\begin{aligned} m_{tan} &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{(h^3+6h^2+12h+8)-8}{h} \\ &= \lim_{h\to 0}\frac{h^3+6h^2+12h}{h} \\ &= \lim_{h\to 0}(h^2+6h+12) \\ &= 12. \end{aligned}$$

Thus the slope of the tangent line is 12. Using the point-slope formula above,

$$y-8 = 12(x-2)$$

or

$$y = 12x-16$$

Next we are interested in finding a formula for the slope of the tangent line at *any point* on the curve f. Such a formula would be the same formula that we are using except we replace the constant x_0 by the variable x. This yields

$$m_{tan} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

We denote this formula by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

where $f(x)$ is read "f prime of x." The next example illustrate its usefulness.

Example 2:

If $f(x) = x^2 - 3$, find $f(x)$ and use the result to find the slope of the tangent line at $x = 2$ and $x = -1$.

Solution:

Since

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{[(x+h)^2 - 3] - [x^2 - 3]}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3 - x^2 + 3}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

To find the slope, we simply substitute $x = 2$ into the result $f(x)$,

$$\begin{aligned} f'(x) &= 2x \\ f'(2) &= 2(2) \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} f'(x) &= 2x \\ f'(-1) &= 2(-1) \\ &= -2 \end{aligned}$$

Thus slopes of the tangent lines at $x = 2$ and $x = -1$ are 4 and -2, respectively.

Example 3:

Find the slope of the tangent line to the curve $y = \frac{1}{x}$ that passes through the point $(1, 1)$.

Solution:

Using the slope of the tangent formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and substituting $y = \frac{1}{x}$,

$$\begin{aligned}
y' &= \lim_{h\to 0} \frac{\left(\frac{1}{x+h}\right) - \frac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{x-x-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2}.
\end{aligned}$$

Substituting $x = 1$,

$$\begin{aligned}
y' &= \frac{-1}{1} \\
&= -1.
\end{aligned}$$

Thus the slope of the tangent line at $x = 1$ for the curve $y = \frac{1}{x}$ is $m = -1$. To find the equation of the tangent line, we simply use the point-slope formula,

$$y - y_0 = m(x - x_0),$$

where $(x_0, y_0) = (1, 1)$.

$$\begin{aligned}
y - 1 &= -1(x-1) \\
&= -x + 1 + 1 \\
&= -x + 2,
\end{aligned}$$

which is the equation of the tangent line.

Average Rates of Change

The primary concept of calculus involves calculating the rate of change of a quantity with respect to another. For example, speed is defined as the rate of change of the distance travelled with respect to time. If a person travels 120 miles in four hours, his speed is $120/4 = 30$ mi/hr. This speed is called the *average speed* or the *average rate of change* of distance with respect to time. Of course the person who travels 120 miles at a rate of 30 mi/hr for four hours does not do so continuously. He must have slowed down or sped up during the four-hour period. But it does suffice to say that he traveled for four hours at an average rate of 30 miles per hour. However, if the driver strikes a tree, it would not be his average speed that determines his survival but his speed at the *instant of the collision*. Similarly, when a bullet strikes a target, it is not the average speed that is significant but its *instantaneous speed* at the moment it strikes. So here we have two distinct kinds of speeds, average speed and instantaneous speed.

The average speed of an object is defined as the object's displacement $\triangle x$ divided by the time interval $\triangle t$ during which the displacement occurs:

$$v = \frac{\triangle x}{\triangle t} = \frac{x_1 - x_0}{t_1 - t_0}.$$

Notice that the points (t_0, x_0) and (t_1, x_1) lie on the position-versus-time curve, as Figure 1 shows. This expression is also the expression for the slope of a secant line connecting the two points. Thus we conclude that the average velocity of an object between time t_0 and t_1 is represented geometrically by the slope of the secant line connecting the two points (t_0, x_0) and (t_1, x_1). If we choose t_1 close to t_0, then the average velocity will closely approximate the instantaneous velocity at time t_0.

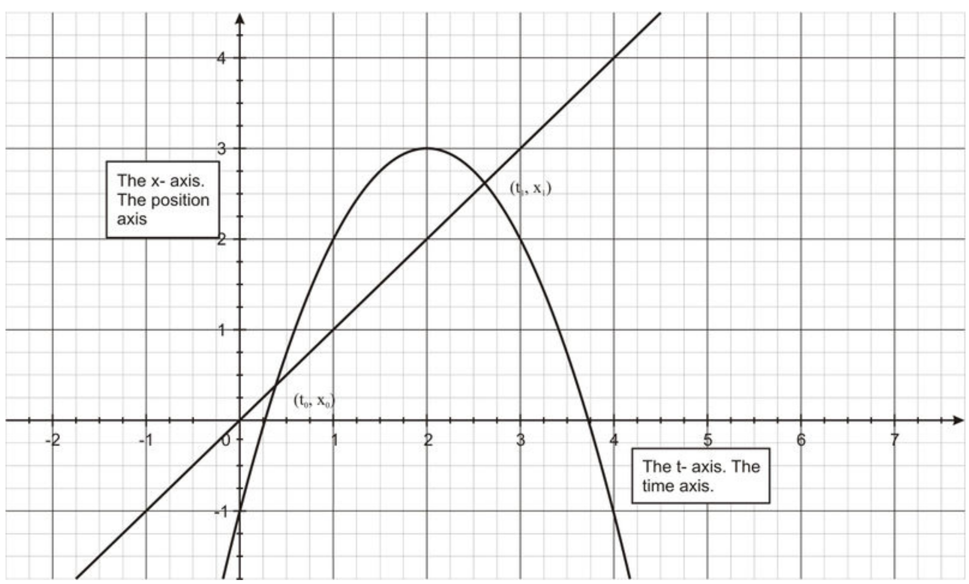


Figure 1

Geometrically, the average rate of change is represented by the slope of a secant line and the instantaneous rate of change is represented by the slope of the tangent line (Figures 2 and 3).

Average Rate of Change (such as the *average velocity*)

The average rate of change of $x = f(t)$ over the time interval $[t_0, t_1]$ is the slope m_{sec} of the secant line to the points $(t_0, f(t_0))$ and $(t_1, f(t_1))$ on the graph (Figure 2).

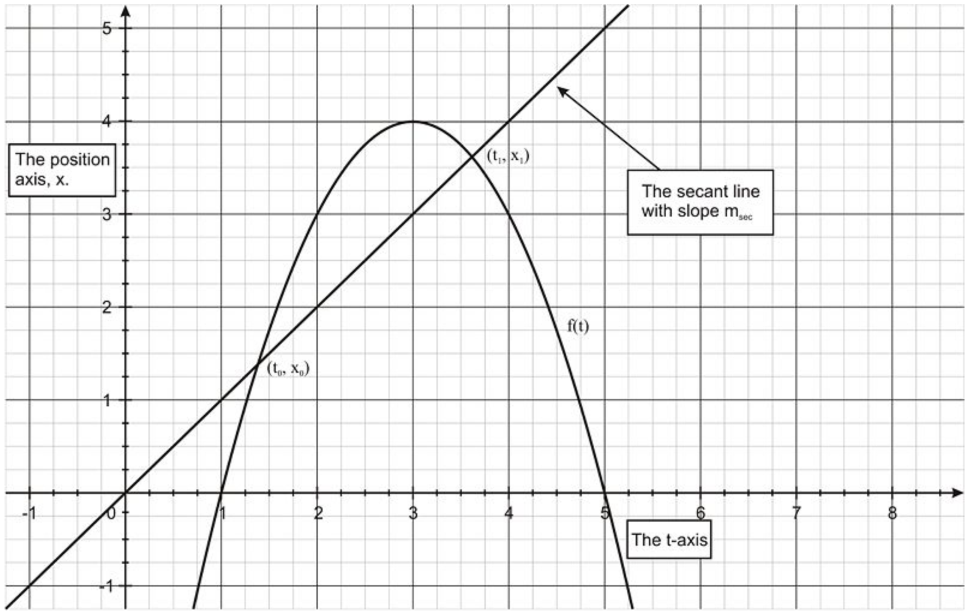


Figure 2

$$m_{sec} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

Instantaneous Rate of Change

The instantaneous rate of change of $x = f(t)$ at the time t_0 is the slope m_{tan} of the tangent line at the time t_0 on the graph.

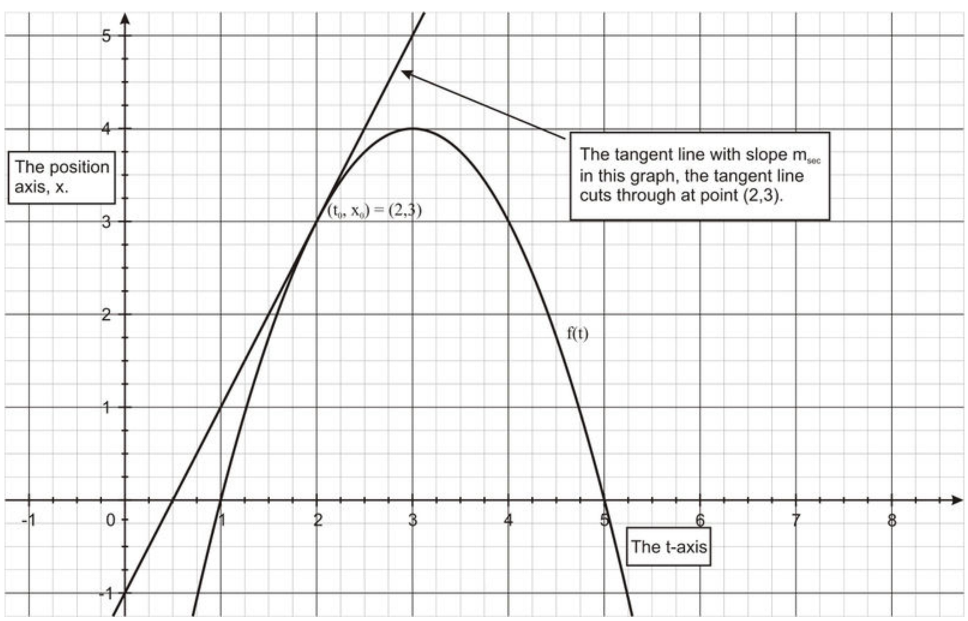


Figure 3

$$m_{tan} = f'(t_0) = \lim_{t_1 \to t_0} \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

Example 4:

Suppose that $y = x^2 - 3$.

1. Find the average rate of change of y with respect to x over the interval $[0,2]$.
2. Find the instantaneous rate of change of y with respect to x at the point $x = -1$.

Solution:

1. Applying the formula for Average Rate of Change with $f(x) = x^2 - 3$ and $x_0 = 0$ and $x_1 = 2$ yields

$$\begin{aligned} m_{sec} &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= \frac{f(2) - f(0)}{2 - 0} \\ &= \frac{1 - (-3)}{2} \\ &= 2 \end{aligned}$$

This means that the average rate of change of y is 2 units per unit increase in x over the interval $[0,2]$.

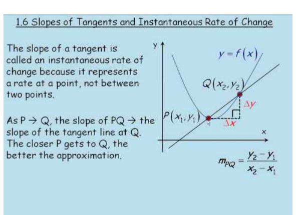
2. From the example above, we found that $f'(x) = 2x$, so

$$\begin{aligned} m_{tan} &= f'(x_0) \\ &= f'(-1) \\ &= 2(-1) \\ &= -2 \end{aligned}$$

This means that the instantaneous rate of change is negative. That is, y is decreasing at $x = -1$. It is decreasing at a rate of 2 units per unit increase in x.

Multimedia Links

For a video explaining instantaneous rates of change **(4.2)**, see Slopes of Tangents and Instantaneous Rates of Ch ange (9:26).



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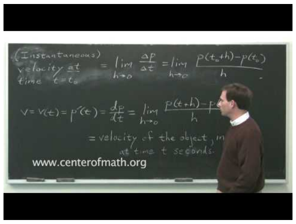
For a video with an application regarding velocity **(4.2)**, see Calculus Help: Instantaneous Rates of Change (9:03).



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The following applet illustrates how the slope of a secant line can become the slope of the tangent to a curve at a point as $h \to 0$. Follow the directions on the page to explore how changing the distance between the two points makes the slope of the secant approach the slope of the tangent Slope at a Point Applet . Note that the function and the point of tangency can also be edited in this simulator.

Review Questions

1. Given the function $y = 1/2\,x^2$ and the values of $x_0 = 3$ and $x_1 = 4$, find
   a. The average rate of change of y with respect to x over the interval $[x_0, x_1]$.
   b. The instantaneous rate of change of y with respect to x at x_0.
   c. The slope of the tangent line at x_1.
   d. The slope of the secant line between points x_0 and x_1.
   e. Make a sketch of $y = 1/2\,x^2$ and show the secant and tangent lines at their respective points.
2. Repeat problem #1 for $f(x) = 1/x$ and the values $x_0 = 2$ and $x_1 = 3$.
3. Find the slope of the graph $f(x) = x^2 + 1$ at a general point x. What is the slope of the tangent line at $x_0 = 6$?
4. Suppose that $y = 1/\sqrt{x}$.
   a. Find the average rate of change of y with respect to x over the interval $[1, 3]$.
   b. Find the instantaneous rate of change of y with respect to x at point $x = 1$.
5. A rocket is propelled upward and reaches a height (in meters) of $h(t) = 4.9t^2$ in t seconds.
   a. How high does it reach in 35 seconds?
   b. What is the average velocity of the rocket during the first 35 seconds?
   c. What is the average velocity of the rocket during the first 200 meters?
   d. What is the instantaneous velocity of the rocket at the end of the 35 seconds?
6. A particle moves in the positive direction along a straight line so that after t nanoseconds, its traversed distance is given by $\chi(t) = 9.9t^3$ nanometers.
   a. What is the average velocity of the particle during the first 2 nanoseconds?
   b. What is the instantaneous velocity of the particle at $t = 2$ nanoseconds?

2.2 The Derivative

Learning Objectives

A student will be able to:

- Demonstrate an understanding of the derivative of a function as a slope of the tangent line.
- Demonstrate an understanding of the derivative as an instantaneous rate of change.
- Understand the relationship between continuity and differentiability.

The function $f'(x)$ that we defined in the previous section is so important that it has its own name.

The Derivative

The function f' is defined by the new function

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

where f ***is called the*** *derivative* ***of*** f ***with respect to*** x. ***The domain of*** f consists of all the values of x for which the limit exists.

Based on the discussion in previous section, the derivative f' represents the slope of the tangent line at point x. Another way of interpreting it is to say that the function $y=f(x)$ has a derivative f' whose value at x is the instantaneous rate of change of y with respect to point x.

Example 1:

Find the derivative of $f(x)=\frac{x}{x+1}$.

Solution:

We begin with the definition of the derivative,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left[f(x+h)-f(x)\right],$$

where

$$f(x)=\frac{x}{x+1}$$
$$f(x+h)=\frac{x+h}{x+h+1}$$

Substituting into the derivative formula,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h}{x+h+1} - \frac{x}{x+1}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2+x+hx+h-x^2-xh-x}{(x+h+1)(x+1)}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{h}{(x+h+1)(x+1)}\right] \\
&= \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} \\
&= \frac{1}{(x+1)^2}.
\end{aligned}$$

Example 2:

Find the derivative of $f(x) = \sqrt{x}$ and the equation of the tangent line at $x_0 = 1$.

Solution:

Using the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{1}{h} \frac{x+h-x}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Thus the slope of the tangent line at $x_0 = 1$ is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

For $x_0 = 1$, we can find y_0 by simply substituting into $f(x)$.

$$\begin{aligned}
f(x_0) &\equiv y_0 \\
f(1) &= \sqrt{1} = 1 \\
y_0 &= 1.
\end{aligned}$$

Thus the equation of the tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \frac{1}{2}(x - 1)$$
$$y = \frac{1}{2}x + \frac{1}{2}.$$

Notation

Calculus, just like all branches of mathematics, is rich with notation. There are many ways to denote the derivative of a function $y = f(x)$ in addition to the most popular one, $f'(x)$. They are:

$$f'(x) \qquad \frac{dy}{dx} \qquad y' \qquad \frac{df}{dx} \qquad \frac{df(x)}{dx}$$

In addition, when substituting the point x_0 into the derivative we denote the substitution by one of the following notations:

$$f'(x_0) \qquad \frac{dy}{dx} \mid x - x_0 \qquad \left.\frac{df}{dx}\right|x - x_0 \qquad \frac{df(x_0)}{dx}$$

Existence and Differentiability of a Function

If, at the point $(x_0, f(x_0))$, the limit of the slope of the secant line does not exist, then the derivative of the function $f(x)$ at this point does not exist either. That is,

if

$$m_{sec} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \text{Does not exist}$$

then the derivative $f'(x)$ also fails to exist as $x \to x_0$. The following examples show four cases where the derivative fails to exist.

1. At a *corner*. For example $f(x) = |x|$, where the derivative on both sides of $x = 0$ differ (Figure 4).
2. At a *cusp*. For example $f(x) = x^{2/3}$, where the slopes of the secant lines approach $+\infty$ on the right and $-\infty$ on the left (Figure 5).
3. A *vertical tangent*. For example $f(x) = x^{1/3}$, where the slopes of the secant lines approach $+\infty$ on the right and $-\infty$ on the left (Figure 6).
4. A *jump discontinuity*. For example, the step function (Figure 7)

$$f(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

where the limit from the left is -2 and the limit from the right is 2.

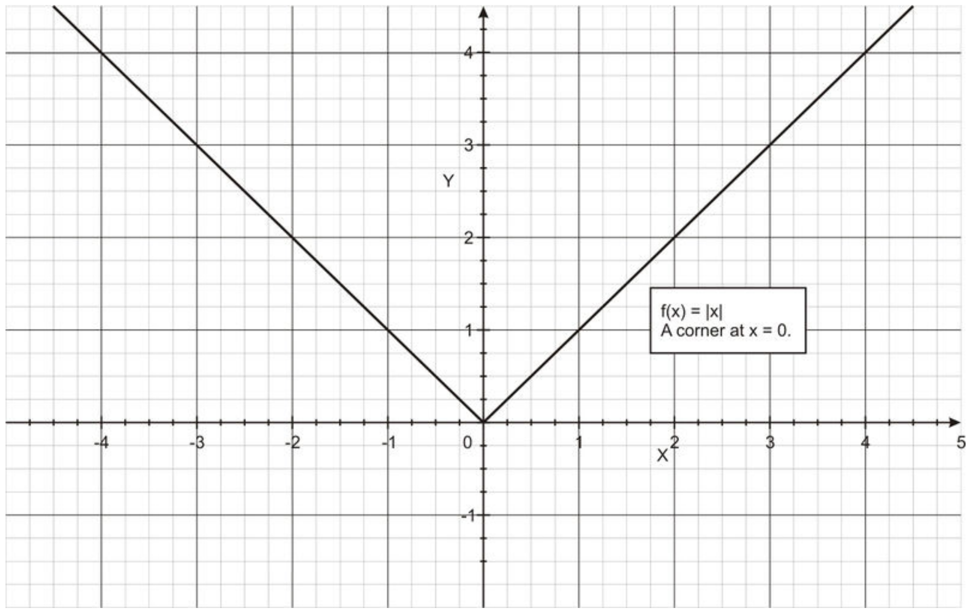


Figure 4

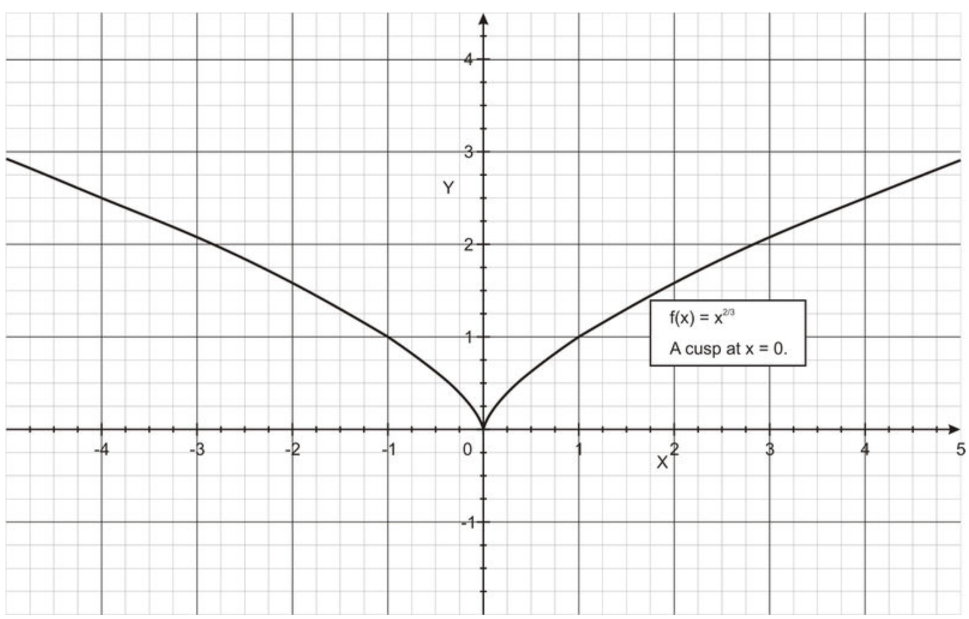


Figure 5

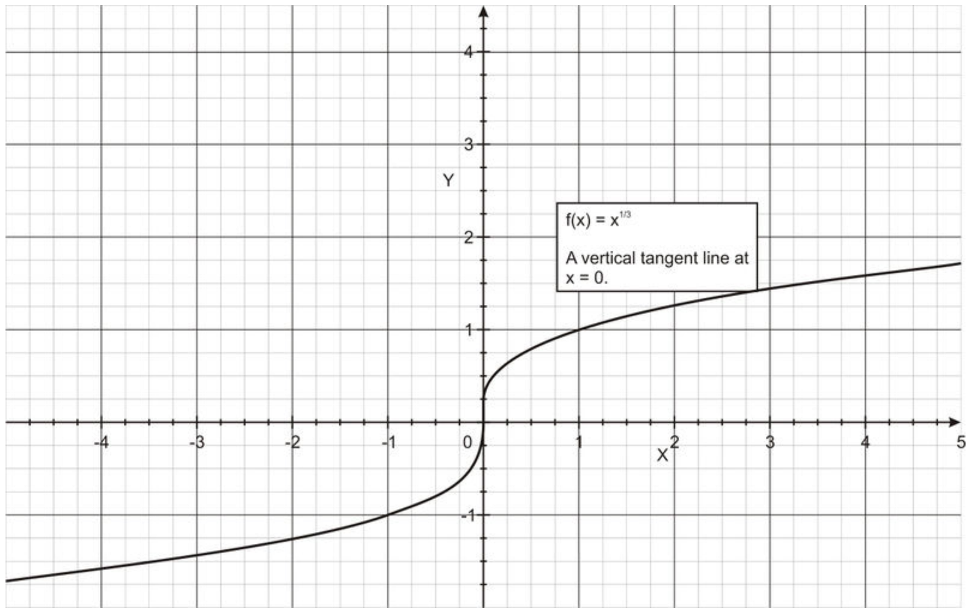


Figure 6

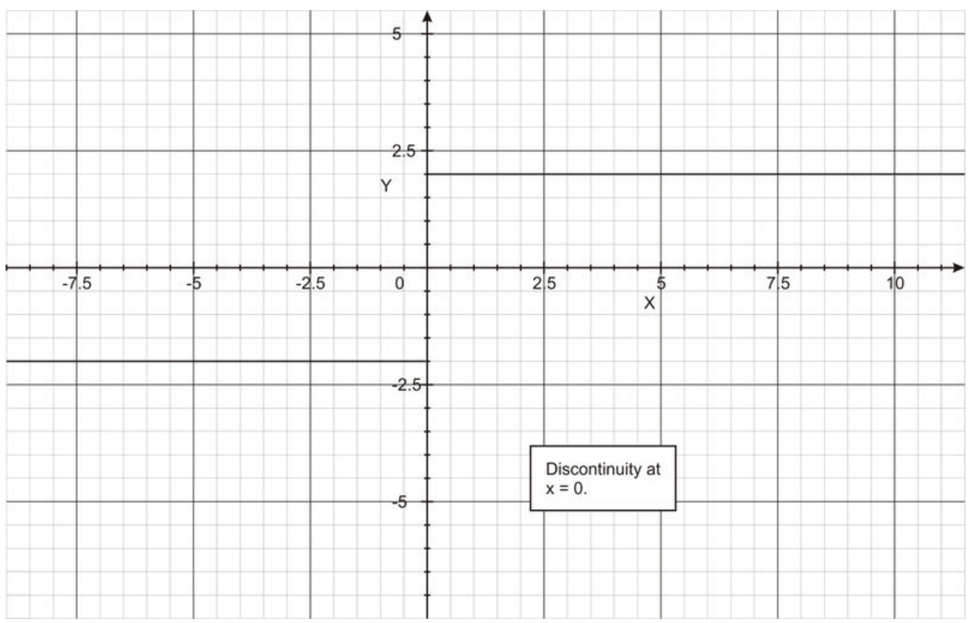


Figure 7

Many functions in mathematics do not have corners, cusps, vertical tangents, or jump discontinuities. We call them **differentiable functions**.

From what we have learned already about differentiability, it will not be difficult to show that continuity is an important condition for differentiability. The following theorem is one of the most important theorems in calculus:

Differentiability and Continuity

If f is differentiable at x_0, then f is also continuous at x_0.

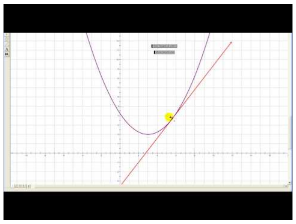
The logically equivalent statement is quite useful: If f is *not* continuous at x_0, then f is not differentiable at x_0.

(The converse is not necessarily true.)

We have already seen that the converse is not true in some cases. The function can have a cusp, a corner, or a vertical tangent and still be continuous, but it is not differentiable.

Multimedia Links

For an introduction to the derivative **(4.0)(4.1)**, see Math Video Tutorials by James Sousa, Introduction to the Derivative (9:57).



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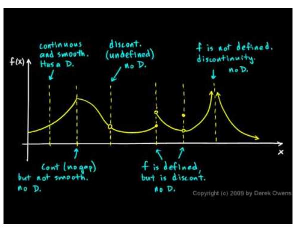
The following simulator traces the instantaneous slope of a curve and graphs a qualitative form of derivative function on an axis below the curve Surfing the Derivative .

The following applet allows you to explore the relationship between a function and its derivative on a graph. Notice that as you move x along the curve, the slope of the tangent line to $f(x)$ is the height of the derivative function, $f'(x)$Derivative Applet . This applet is customizable–after doing the steps outlined on the page, feel free to change the function definition and explore the derivative of many functions.

For a video presentation of differentiability and continuity **(4.3)**, see Differentiability and Continuity (6:31).



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Review Questions

In problems 1 through 6, use the definition of the derivative to find $f'(x)$ and then find the equation of the tangent line at $x = x_0$.

1. $f(x) = 6x^2; x_0 = 3$
2. $f(x) = \sqrt{x+2}; x_0 = 8$
3. $f(x) = 3x^3 - 2; x_0 = -1$
4. $f(x) = \frac{1}{x+2}; x_0 = -1$
5. $f(x) = ax^2 - b$, (where a and b are constants); $x_0 = b$
6. $f(x) = x^{1/3}; x_0 = 1$.
7. Find $dy/dx|_{x=1}$ given that $y = 5x^2 - 2$.
8. Show that $f(x) = \sqrt[3]{x}$ is defined at $x = 0$ but it is not differentiable at $x = 0$. Sketch the graph.
9. Show that

$$f(x) = \begin{cases} x^2 + 1 & x \geq 1 \\ 2x & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Hint: Take the limit from both sides. Sketch the graph of f.

2.3 Techniques of Differentiation

Learning Objectives

A student will be able to:

- Use various techniques of differentiations to find the derivatives of various functions.
- Compute derivatives of higher orders.

Up to now, we have been calculating derivatives by using the definition. In this section, we will develop formulas and theorems that will calculate derivatives in more efficient and quick ways. It is highly recommended that you become very familiar with all of these techniques.

The Derivative of a Constant

If $f(x) = c$ where c is a constant, then $f'(x) = 0$.

In other words, the derivative or slope of any constant function is zero.

Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

Example 1:

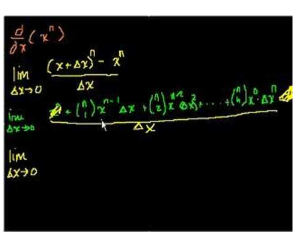
If $f(x) = 16$ for all x, then $f'(x) = 0$ for all x. We can also write $d/dx(16) = 0$.

The Power Rule

If n is a positive integer, then for all real values of x

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

The proof of the power rule is omitted in this text, but it is available at http://en.wikipedia.org/wiki/Calculus_with_polynomials and also in video form at Khan Academy Proof of the Power Rule .



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Note that this proof depends on using the binomial theorem from precalculus.

Example 2:

If $f(x) = x^3$, then

$$f'(x) = 3x^{3-1} = 3x^2$$

and

$$\begin{aligned}
\frac{d}{dx}[x] &= 1 \cdot x^{1-1} = x^0 = 1,\\
\frac{d}{dx}[\sqrt{x}] &= \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}},\\
\frac{d}{dx}\left[\frac{1}{x^3}\right] &= \frac{d}{dx}[x^{-3}] = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4},
\end{aligned}$$

The Power Rule and a Constant

If c is a constant and f is differentiable at all x, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].$$

In simpler notation,

$$(cf)' = c(f)' \cdot cf'.$$

In other words, the derivative of a constant times a function is equal to the constant times the derivative of the function.

Example 3:

$$\frac{d}{dx}[4x^3] = 4\frac{d}{dx}[x^3] = 4[3x^2] = 12x^2.$$

Example 4:

$$\frac{d}{dx}\left[\frac{-2}{x^4}\right] = \frac{d}{dx}[-2x^{-4}] = -2\frac{d}{dx}[x^{-4}] = -2[-4x^{-4-1}] = -2[-4x^{-5}] = 8x^{-5} = \frac{8}{x^5}.$$

Derivatives of Sums and Differences

If f and g are two differentiable functions at x, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

and

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$$

In simpler notation,

$$(f \pm g)' = f' \pm g'.$$

Example 5:

$$\begin{aligned}\frac{d}{dx}[3x^2+2x] &= \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x]\\ &= 3\frac{d}{dx}[x^2] + 2\frac{d}{dx}[x]\\ &= 3[2x]+2[1]\\ &= 6x+2.\end{aligned}$$

Example 6:

$$\begin{aligned}\frac{d}{dx}[x^3-5x^2] &= \frac{d}{dx}[x^3] - 5\frac{d}{dx}[x^2]\\ &= 3x^2-5[2x]\\ &= 3x^2-10x.\end{aligned}$$

The Product Rule

If f and g are differentiable at x, then

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).$$

In a simpler notation,

$$(f\cdot g)' = f\cdot g' + g\cdot f'.$$

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Keep in mind that

$$(f\cdot g)' \neq f'\cdot g'.$$

Example 7:

Find $\frac{dy}{dx}$ for $y = (3x^4 + 2)(7x^3 - 1)$.

Solution:

There are two methods to solve this problem. One is to multiply the product and then use the derivative of the sum rule. The second is to directly use the product rule. Either rule will produce the same answer. We begin with the sum rule.

$$\begin{aligned} y &= (3x^4 + 2)(7x^3 - 1) \\ &= 21x^7 - 3x^4 + 14x^3 - 2. \end{aligned}$$

Taking the derivative of the sum yields

$$\begin{aligned} \frac{dy}{dx} &= 147x^6 - 12x^3 + 42x^2 + 0 \\ &= 147x^6 - 12x^3 + 42x^2. \end{aligned}$$

Now we use the product rule,

$$\begin{aligned} \frac{dy}{dx} &= (3x^4 + 2) \cdot (7x^3 - 1)' + (3x^4 + 2)' \cdot (7x^3 - 1) \\ &= (3x^4 + 2)(21x^2) + (12x^3)(7x^3 - 1) \\ &= (63x^6 + 42x^2) + (84x^6 - 12x^3) \\ &= 147x^6 - 12x^3 + 42x^2, \end{aligned}$$

which is the same answer.

The Quotient Rule

If f and g are differentiable functions at x and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

In simpler notation,

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}.$$

The derivative of a quotient of two functions is the bottom times the derivative of the top minus the top times the derivative of the bottom all over the bottom squared.

Keep in mind that the order of operations is important (because of the minus sign in the numerator) and

$$\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}.$$

Example 8:

Find dy/dx for

$$y = \frac{x^2 - 5}{x^3 + 2}$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[\frac{x^2-5}{x^3+2}\right] \\
&= \frac{(x^3+2)\cdot(x^2-5)' - (x^2-5)\cdot(x^3+2)'}{(x^3+2)^2} \\
&= \frac{(x^3+2)(2x) - (x^2-5)(3x^2)}{(x^3+2)^2} \\
&= \frac{2x^4 + 4x - 3x^4 + 15x^2}{(x^3+2)^2} \\
&= \frac{-x^4 + 15x^2 + 4x}{(x^3+2)^2} \\
&= \frac{x(-x^3 + 15x + 4)}{(x^3+2)^2}.
\end{aligned}$$

Example 9:

At which point(s) does the graph of $y = \frac{x}{x^2+9}$ have a horizontal tangent line?

Solution:

Since the slope of a horizontal line is zero, and since the derivative of a function signifies the slope of the tangent line, then taking the derivative and equating it to zero will enable us to find the points at which the slope of the tangent line equals to zero, i.e., the locations of the horizontal tangents.

$$\begin{aligned}
y &= \frac{x}{x^2+9}, \\
y' &= \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = 0.
\end{aligned}$$

Multiplying by the denominator and solving for x,

$$\begin{aligned}
x^2 + 9 - 2x^2 &= 0 \\
x^2 &= 9 \\
x &= \pm 3.
\end{aligned}$$

Therefore the tangent line is horizontal at $x = -3, +3$.

Higher Derivatives

If the derivative f' ***of the function*** f ***is differentiable, then the derivative of*** f', denoted by f'', is called the **second derivative** of f. We can continue the process of differentiating derivatives and obtain the third, fourth, fifth, and even higher derivatives of f. They are denoted by f''', $f^{(4)}$, $f^{(5)}$, etc.

The second derivative, f'', can also be written as $\frac{d^2y}{dx^2}$, and f''' can be written as $\frac{d^3y}{dx^3}$. For still higher derivatives, $f^{(n)} = \frac{d^nx}{dx^n}$.

Example 10:

Find the fifth derivative of $f(x) = 2x^4 - 3x^3 + 5x^2 - x - 1$.

Solution:

$$\begin{aligned}
f'(x) &= 8x^3 - 9x^2 + 10x - 1 \\
f''(x) &= 24x^2 - 18x + 10 \\
f'''(x) &= 48x - 18 \\
f^{(4)}(x) &= 48 \\
f^{(5)}(x) &= 0
\end{aligned}$$

Example 11:

Show that $y = x^3 + 3x + 2$ satisfies the differential equation $y''' + xy'' - 2y' = 0$.

Solution:

We need to obtain the first, second, and third derivatives and substitute them into the differential equation.

$$\begin{aligned}
y &= x^3 + 3x + 2 \\
y' &= 3x^2 + 3 \\
y'' &= 6x \\
y''' &= 6.
\end{aligned}$$

Substituting,

$$\begin{aligned}
y''' + xy'' - 2y' &= 6 + x(6x) - 2(3x^2 + 3) \\
&= 6 + 6x^2 - 6x^2 - 6 \\
&= 0
\end{aligned}$$

which satisfies the equation.

Review Questions

Use the results of this section to find the derivatives dy/dx.

1. $y = 5x^7$
2. $y = \frac{1}{2}(x^3 - 2x^2 + 1)$
3. $y = \sqrt{2}x^3 - \frac{1}{\sqrt{2}}x^2 + 2x + \sqrt{2}$
4. $y = a^2 - b^2 + x^2 - a - b + x$ (where a, b are constants)
5. $y = x^{-3} + \frac{1}{x^7}$

6. $y = (x^3 - 3x^2 + x)(2x^3 + 7x^4)$
7. $y = (\frac{1}{x} + \frac{1}{x^2})(3x^4 - 7)$
8. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$
9. $y = \frac{3}{\sqrt{x}+3}$
10. $y = \frac{4x+1}{x^2-9}$
11. Newton's Law of Universal Gravitation states that the gravitational force between two masses (say, the earth and the moon), m and M, is equal to their product divided by the square of the distance r between them. Mathematically,

$$F = G\frac{mM}{r^2},$$

where G is the Universal Gravitational Constant $\left(1.602 \times 10^{-11} \frac{Nm^2}{kg^2}\right)$. If the distance r between the two masses is changing, find a formula for the instantaneous rate of change of F with respect to the separation distance r.
12. Find

$$\frac{d}{d\psi}\left[\frac{\psi\psi_0 + \psi^3}{3 - \psi_0}\right]$$

where ψ_0 is a constant.
13. Find $\left.\frac{d^3y}{dx^3}\right|_{x=1}$, where $y = \frac{2}{x^3}$.

2.4 Derivatives of Trigonometric Functions

Learning Objectives

A student will be able to:

- Compute the derivatives of various trigonometric functions.

If the angle h is measured in radians,

$\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$.

We can use these limits to find an expression for the derivative of the six trigonometric functions $\sin x, \cos x, \tan x, \sec x, \csc x$, and $\cot x$. We first consider the problem of differentiating $\sin x$, using the definition of the derivative.

$$\frac{d}{dx}[\sin x] = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Since

$$\sin(\alpha+\beta) = \sin\alpha \ \cos\beta + \cos\alpha \ \sin\beta.$$

The derivative becomes

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\
&= -\sin x \cdot \lim_{h\to 0} \left(\frac{1-\cos h}{h}\right) + \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h}\right) \\
&= -\sin x \cdot (0) + \cos x \cdot (1) \\
&= \cos x.
\end{aligned}$$

Therefore,

$$\frac{d}{dx}[\sin x] = \cos x.$$

It will be left as an exercise to prove that

$$\frac{d}{dx}[\cos x] = -\sin x.$$

The derivatives of the remaining trigonometric functions are shown in the table below.

Derivatives of Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \cos x\\
\frac{d}{dx}[\cos x] &= -\sin x\\
\frac{d}{dx}[\tan x] &= \sec^2 x\\
\frac{d}{dx}[\sec x] &= \sec x \tan x\\
\frac{d}{dx}[\csc x] &= -\csc x \cot x\\
\frac{d}{dx}[\cot x] &= -\csc^2 x
\end{aligned}$$

Keep in mind that for all the derivative formulas for the trigonometric functions, the argument x is measured in radians.

Example 1:

Show that $\frac{d}{dx}[\tan x] = \sec^2 x$.

Solution:

It is possible to prove this relation by the definition of the derivative. However, we use a simpler method.

Since

$$\tan x = \frac{\sin x}{\cos x},$$

then

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

Using the quotient rule,

$$\begin{aligned}
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}\\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x}\\
&= \sec^2 x
\end{aligned}$$

Example 2:

Find $f'(x)$ if $f(x) = x^2 \cos x + \sin x$.

Solution:

Using the product rule and the formulas above, we obtain

$$\begin{aligned} f'(x) &= x^2(-\sin x) + 2x\cos x + \cos x \\ &= -x^2 \sin x + 2x\cos x + \cos x. \end{aligned}$$

Example 3:

Find dy/dx if $y = \frac{\cos x}{1-\tan x}$. What is the slope of the tangent line at $x = \pi/3$?

Solution:

Using the quotient rule and the formulas above, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{(1-\tan x)(-\sin x) - (\cos x)(-\sec^2 x)}{(1-\tan x)^2} \\ &= \frac{-\sin x + \tan x \sin x + \cos x \sec^2 x}{(1-\tan x)^2} \\ &= \frac{-\sin x + \tan x \sin x + \sec x}{(1-\tan x)^2} \end{aligned}$$

To calculate the slope of the tangent line, we simply substitute $x = \pi/3$:

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = \frac{-\sin(\frac{\pi}{3}) + \tan(\frac{\pi}{3})\sin(\frac{\pi}{3}) + \sec(\frac{\pi}{3})}{(1-\tan(\frac{\pi}{3}))^2}.$$

We finally get the slope to be approximately

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 4.9.$$

Example 4:

If $y = \sec x$, find $y''(\pi/3)$.

Solution:

$$\begin{aligned} y' &= \sec x \tan x \\ y'' &= \sec x(\sec^2 x) + (\sec x \tan x)\tan x \\ &= \sec^3 x + \sec x \tan^2 x. \end{aligned}$$

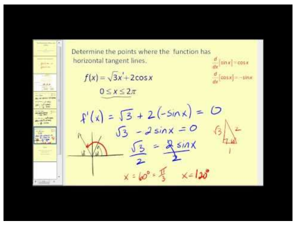
Substituting for $x = \pi/3$,

$$\begin{aligned} y'' &= \sec^3\left(\frac{\pi}{3}\right) + \sec\left(\frac{\pi}{3}\right)\tan^2\left(\frac{\pi}{3}\right) \\ &= (2)^3 + (2)(\sqrt{3})^2 \\ &= 8 + (2)(3) \\ &= 14. \end{aligned}$$

Thus $y''(\pi/3) = 14$.

Multimedia Links

For examples of finding the derivatives of trigonometric functions **(4.4)**, see Math Video Tutorials by James Sousa, The Derivative of Sine and Cosine (9:21).



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Review Questions

Find the derivative y' of the following functions:

1. $y = x\sin x + 2$
2. $y = x^2 \cos x - x\tan x - 1$
3. $y = \sin^2 x$
4. $y = \frac{\sin x - 1}{\sin x + 1}$
5. $y = \frac{\cos x + \sin x}{\cos x - \sin x}$
6. $y = \frac{\sqrt{x}}{\tan x} + 2$
7. $y = \csc x \sin x + x$
8. $y = \frac{\sec x}{\csc x}$
9. If $y = \csc x$, find $y''(\pi/6)$.
10. Use the definition of the derivative to prove that $\frac{d}{dx}[\cos x] = -\sin x$.

2.5 The Chain Rule

Learning Objectives

A student will be able to:

- Know the chain rule and its proof.
- Apply the chain rule to the calculation of the derivative of a variety of composite functions.

We want to derive a rule for the derivative of a composite function of the form $f \circ g$ in terms of the derivatives of f and g. This rule allows us to differentiate complicated functions in terms of known derivatives of simpler functions.

The Chain Rule

If g is a differentiable function at x and f is differentiable at $g(x)$, then the composition function $f \circ g = f(g(x))$ is differentiable at x. The derivative of the composite function is:

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Another way of expressing, if $u = u(x)$ and $f = f(u)$, then

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}.$$

And a final way of expressing the chain rule is the easiest form to remember: If y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 1:

Differentiate $f(x) = (2x^3 - 4x^2 + 5)^2$.

Solution:

Using the chain rule, let $u = 2x^3 - 4x^2 + 5$. Then

$$\begin{aligned}\frac{d}{dx}[(2x^2 - 4x^2 + 5)^2] &= \frac{d}{dx}[u^2] \\ &= 2u\frac{du}{dx} \\ &= 2(2x^3 - 4x^2 + 5)(6x^2 - 8x).\end{aligned}$$

The example above is one of the most common types of composite functions. It is a power function of the type

$$y = [u(x)]^n.$$

The rule for differentiating such functions is called the **General Power Rule.** It is a special case of the Chain Rule.

The General Power Rule

if

$$y = [u(x)]^n$$

then

$$\frac{dy}{dx} = n[u(x)]^{n-1}u'(x).$$

In simpler form, if

$$y = u^n$$

then

$$y' = nu^{n-1} \cdot u'.$$

Example 2:

What is the slope of the tangent line to the function $y = \sqrt{x^2 - 3x + 2}$ that passes through point $x = 3$?

Solution:

We can write $y = (x^2 - 3x + 2)^{1/2}$. This example illustrates the point that n can be any real number including fractions. Using the General Power Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}(x^2 - 3x + 2)^{\frac{1}{2}-1}(2x - 3) \\
&= \frac{1}{2}(x^2 - 3x + 2)^{-1/2}(2x - 3) \\
&= \frac{(2x - 3)}{2\sqrt{x^2 - 3x + 2}}
\end{aligned}$$

To find the slope of the tangent line, we simply substitute $x = 3$ into the derivative:

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{2(3) - 3}{2\sqrt{3^2 - 3(3) + 2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}.$$

Example 3:

Find dy/dx for $y = \sin^3 x$.

Solution:

The function can be written as $y = [\sin x]^3$. Thus

$$\begin{aligned}\frac{dy}{dx} &= 3[\sin x]^2[\cos x] \\ &= 3\sin^2 x \cos x\end{aligned}$$

Example 4:

Find dy/dx for $y = 5\cos(3x^2 - 1)$.

Solution:

Let $u = 3x^2 - 1$. By the chain rule,

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

where $f(u) = 5\cos u$. Thus

$$\begin{aligned}\frac{dy}{dx} &= 5(-\sin u)(6x) \\ &= -5(6x)\sin u \\ &= -30x\sin(3x^2 - 1)\end{aligned}$$

Example 5:

Find dy/dx for $y = [\cos(\pi x^2)]^3$.

Solution:

This example applies the chain rule twice because there are several functions embedded within each other.

Let u be the inner function and w be the innermost function.

$$\begin{aligned}y &= (u(w))^3 \\ u(x) &= \cos x \\ w(x) &= \pi x^2.\end{aligned}$$

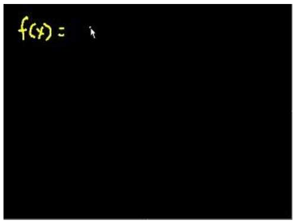
Using the chain rule,

$$\begin{aligned}\frac{d}{dx}[f(u)] &= f'(u)\frac{du}{dx} \\ \frac{d}{dx}[u^3] &= \frac{d}{dx}[\cos^3(\pi x^2)] \\ &= \frac{d}{dx}[\cos(\pi x^2)]^3 \\ &= 3[\cos(\pi x^2)]^2[-\sin(\pi x^2)](2\pi x) \\ &= -6\pi x[\cos(\pi x^2)]^2\sin(\pi x^2).\end{aligned}$$

Notice that we used the General Power Rule and, in the last step, we took the derivative of the argument.

Multimedia Links

For an introduction to the Chain Rule **(5.0)**, see Khan Academy, Calculus: Derivatives 4: The Chain Rule (9:11).



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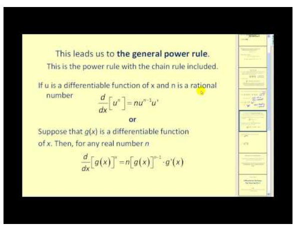
For more examples of the Chain Rule **(5.0)**, see Math Video Tutorials by James Sousa The Chain Rule: Part 1 of 2 (8:45).



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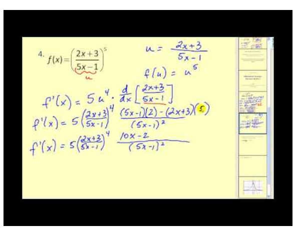
and Math Video Tutorials by James Sousa The Chain Rule: Part 2 of 2 (8:36).



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Review Questions

Find $f'(x)$.

1. $f(x) = (2x^2 - 3x)^{39}$
2. $f(x) = \left(x^3 - \frac{5}{x^2}\right)^{-3}$
3. $f(x) = \frac{1}{\sqrt{3x^2 - 6x + 2}}$
4. $f(x) = \sin^3 x$
5. $f(x) = \sin x^3$
6. $f(x) = \sin^3 x^3$
7. $f(x) = \tan(4x^5)$
8. $f(x) = \sqrt{4x - \sin^2 2x}$
9. $f(x) = \frac{\sin x}{\cos(3x-2)}$
10. $f(x) = (5x+8)^3(x^3+7x)^{13}$

11. $f(x) = \left(\frac{x-3}{2x-5}\right)^3$

2.6 Implicit Differentiation

Learning Objectives

A student will be able to:

- Find the derivative of variety of functions by using the technique of implicit differentiation.

Consider the equation

$$2xy = 1.$$

We want to obtain the derivative dy/dx. One way to do it is to first solve for y,

$$y = \frac{1}{2x},$$

and then project the derivative on both sides,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left[\frac{1}{2x}\right] \\ &= \frac{-1}{2x^2}. \end{aligned}$$

There is another way of finding dy/dx. We can directly differentiate both sides:

$$\frac{d}{dx}[2xy] = \frac{d}{dx}[1].$$

Using the Product Rule on the left-hand side,

$$\begin{aligned} y\frac{d}{dx}[2x] + 2x\frac{d}{dx}[y] &= 0 \\ y[2] + 2x\frac{dy}{dx} &= 0. \end{aligned}$$

Solving for dy/dx,

$$\frac{dy}{dx} = \frac{-2y}{2x} = \frac{-y}{x}.$$

But since $y = \frac{1}{2x}$, substitution gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{x(2x)} \\ &= \frac{-1}{2x^2}.\end{aligned}$$

which agrees with the previous calculations. This second method is called the **implicit differentiation** method. You may wonder and say that the first method is easier and faster and there is no reason for the second method. That's probably true, but consider this function:

$$3y^2 - \cos y = x^3.$$

How would you solve for y? That would be a difficult task. So the method of implicit differentiation sometimes is very useful, especially when it is inconvenient or impossible to solve for y in terms of x. Explicitly defined functions may be written with a direct relationship between two variables with clear independent and dependent variables. Implicitly defined functions or relations connect the variables in a way that makes it impossible to separate the variables into a simple input output relationship. More notes on explicit and implicit functions can be found at http://en.wikipedia.org/wiki/Implicit_function .

Example 1:

Find dy/dx if $3y^2 - \cos y = x^3$.

Solution:

Differentiating both sides with respect to x and then solving for dy/dx,

$$\begin{aligned}\frac{d}{dx}[3y^2 - \cos y] &= \frac{d}{dx}[x^3] \\ 3\frac{d}{dx}[y^2] - \frac{d}{dx}[\cos y] &= 3x^2 \\ 3(2y\frac{dy}{dx}) - (-\sin y)\frac{dy}{dx} &= 3x^2 \\ 6y\frac{dy}{dx} + \sin y\frac{dy}{dx} &= 3x^2\end{aligned}$$

y + $\sin y \frac{dy}{dx} = 3x^2$.

Solving for dy/dx, we finally obtain

$$\frac{dy}{dx} = \frac{3x^2}{6y + \sin y}.$$

Implicit differentiation can be used to calculate the slope of the tangent line as the example below shows.

Example 2:

Find the equation of the tangent line that passes through point $(1,2)$ to the graph of $8y^3 + x^2y - x = 3$.

Solution:

First we need to use implicit differentiation to find dy/dx and then substitute the point $(1,2)$ into the derivative to find slope. Then we will use the equation of the line (either the slope-intercept form or the point-intercept form) to find the equation of the tangent line. Using implicit differentiation,

$$\begin{aligned}\frac{d}{dx}[8y^3+x^2y-x] &= \frac{d}{dx}[3]\\ 24y^2\frac{dy}{dx}+[(x^2)(1)\frac{dy}{dx}+y(2x)]-1 &= 0\\ 24y^2\frac{dy}{dx}+x^2\frac{dy}{dx}+2xy-1 &= 0\end{aligned}$$

y^2 + x^2

$$\begin{aligned}\frac{dy}{dx} &= 1-2xy\\ \frac{dy}{dx} &= \frac{1-2xy}{24y^2+x^2}.\end{aligned}$$

Now, substituting point $(1,2)$ into the derivative to find the slope,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1-2(1)(2)}{24(2)^2+(1)^2}\\ &= \frac{-3}{97}.\end{aligned}$$

So the slope of the tangent line is $-3/97$, which is a very small value. (What does this tell us about the orientation of the tangent line?)

Next we need to find the equation of the tangent line. The slope-intercept form is

$$y = mx+b,$$

where $m=-3/97$ and b is the $y-$intercept. To find it, simply substitute point $(1,2)$ into the line equation and solve for b to find the $y-$intercept.

$$\begin{aligned}2 &= \left(\frac{-3}{97}\right)(1)+b\\ b &= \frac{197}{97}.\end{aligned}$$

Thus the equation of the tangent line is

$$y = \frac{-3}{97}x+\frac{197}{97}.$$

Remark: we could have used the point-slope form $y-y_1=m(x-x_1)$ and obtained the same equation.

Example 3:

Use implicit differentiation to find d^2y/dx^2 if $5x^2 - 4y^2 = 9$. Also find $\left.\frac{d^2y}{dx^2}\right|_{(x,\ y)=(2,\ 3)}$. What does the second derivative represent?

Solution:

$$\frac{d}{dx}[5x^2 - 4y^2] = \frac{d}{dx}[9]$$
$$10x - 8y\frac{dy}{dx} = 0.$$

Solving for dy/dx,

$$\frac{dy}{dx} = \frac{5x}{4y}.$$

Differentiating both sides implicitly again (and using the quotient rule),

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(4y)(5) - (5x)(4dy/dx)}{(4y)^2}\\ &= \frac{20y}{16y^2} - \frac{20x}{16y^2}\frac{dy}{dx}\\ &= \frac{5}{4y} - \frac{5x}{4y^2}\frac{dy}{dx}.\end{aligned}$$

But since $dy/dx = 5x/4y$, we substitute it into the second derivative:

$$\frac{d^2y}{dx^2} = \frac{5}{4y} - \frac{5x}{4y^2}.\frac{5x}{4y}$$
$$\frac{d^2y}{dx^2} = \frac{5}{4y} - \frac{25x^2}{16y^3}.$$

This is the second derivative of y.

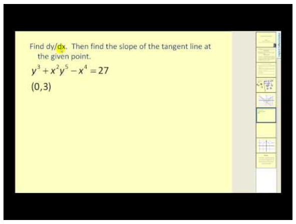
The next step is to find: $\left.\frac{d^2y}{dx^2}\right|_{(x,\ y)=(2,\ 3)}$

$$\begin{aligned}\left.\frac{d^2y}{dx^2}\right|_{(2,\ 3)} &= \frac{5}{4(3)} - \frac{25(2)^2}{16(3)^3}\\ &= \frac{5}{27}.\end{aligned}$$

Since the first derivative of a function represents the rate of change of the function $y = f(x)$ with respect to x, the second derivative represents the rate of change of the rate of change of the function. For example, in kinematics (the study of motion), the speed of an object (y') signifies the change of position with respect to time but acceleration (y'') signifies the rate of change of the speed with respect to time.

Multimedia Links

For more examples of implicit differentiation **(6.0)**, see Math Video Tutorials by James Sousa, Implicit Differenti ation (8:10).



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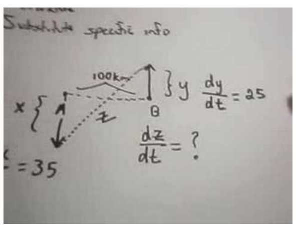
For a video presentation of related rates using implicit differentiation **(6.0)**, see Just Math Tutoring, Related Rates Using Implicit Differentiation (9:56).



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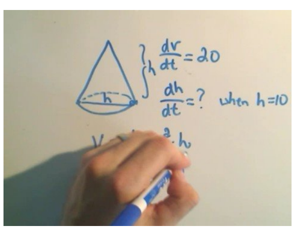
For a presentation of related rates using cones **(6.0)**, see Just Math Tutoring, Related Rates Using Implicit Differe ntiation (2:47).



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Review Questions

Find dy/dx by implicit differentiation.

1. $x^2+y^2=500$
2. $x^2y+3xy-2=1$
3. $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$
4. $\sqrt{x}-\sqrt{y}=\sqrt{3}$
5. $\sin(25xy^2)=x$
6. $\tan^3(x^2-y^2)=\tan(\pi/4)$

In problems #7 and 8, use implicit differentiation to find the slope of the tangent line to the given curve at the specified point.

7. $x^2y - y^2x = 0$ at $(1,1)$
8. $\sin(xy) = y$ at $(\frac{\pi}{2}, 1)$
9. Find y'' by implicit differentiation for $x^3y^3 = 5$.
10. Use implicit differentiation to show that the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is given by $yy_0 = \frac{1}{2}k(x + x_0)$, where k is a constant.

$$\begin{aligned} \frac{d(y^2)}{dx} &= \frac{d(kx)}{dx} \\ 2y\frac{dy}{dx} &= k \\ \frac{dy}{dx} &= \frac{k}{2y} \end{aligned}$$

Second, we substitute y_0 for y, and that gives us the slope m of our tangent line at (x_0, y_0):

$$m = \frac{k}{2y_0}$$

Third, we set up the equation for our tangent line using point-slope form:

$$y - y_0 = \frac{k}{2y_0}(x - x_0)$$

Fourth, and finally, we manipulate this linear equation to get the term yy_0 isolated on the left hand side:

$$\begin{aligned} y - y_0 &= \frac{k}{2y_0}(x - x_0) \\ y &= \frac{k}{2y_0}(x - x_0) + y_0 \\ yy_0 &= \frac{k}{2}(x - x_0) + (y_0)^2 \\ yy_0 &= \frac{k}{2}(x - x_0) + kx_0 \text{ (Using the fact that } y^2 = kx) \\ yy_0 &= \frac{k}{2}(x + x_0) \end{aligned}$$

2.7 Linearization and Newton's Method

Learning Objectives

A student will be able to:

- Approximate a function by the method of linearization.
- Know Newton's Method for approximating roots of a function.

Linearization: The Tangent Line Approximation

If f is a differentiable function at x_0, then the tangent line, $y = mx + b$, to the curve $y = f(x)$ at x_0 is a good approximation to the curve $y = f(x)$ for values of x near x_0 (Figure 8a). If you "zoom in" on the two graphs, $y = f(x)$ and the tangent line, at the point of tangency, $(x_0, f(x_0))$, or if you look at a table of values near the point of tangency, you will notice that the values are very close (Figure 8b).

Since the tangent line passes through point $(x_0, f(x_0))$ and the slope is $f'(x_0)$, we can write the equation of the tangent line, in point-slope form, as

$$y - y_0 = m(x - x_0)$$
$$y - f(x_0) = f'(x_0)(x - x_0)$$

Solving for y,

$$y = f(x_0) + f'(x_0)(x - x_0)$$

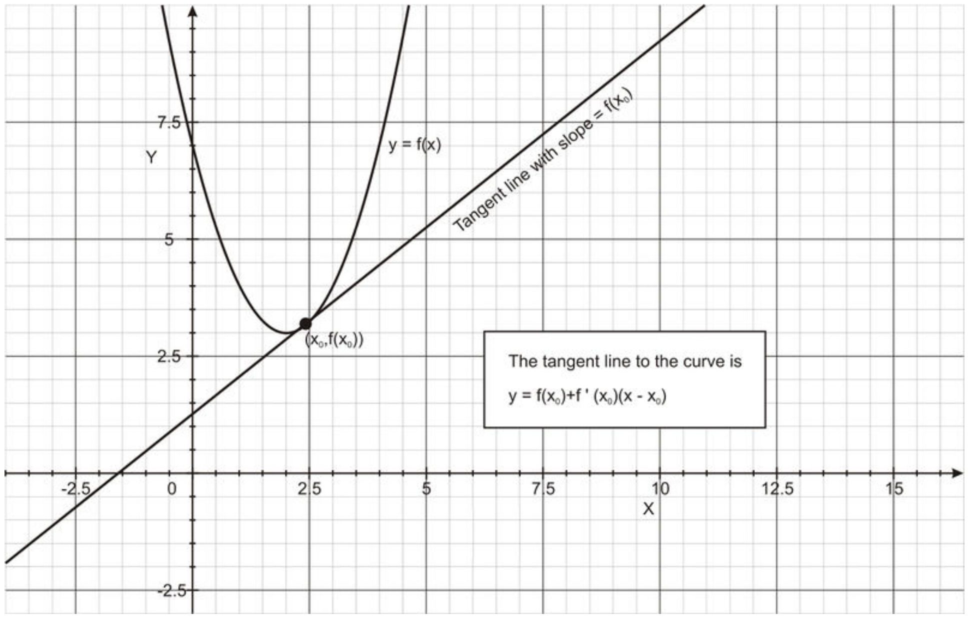


Figure 8a

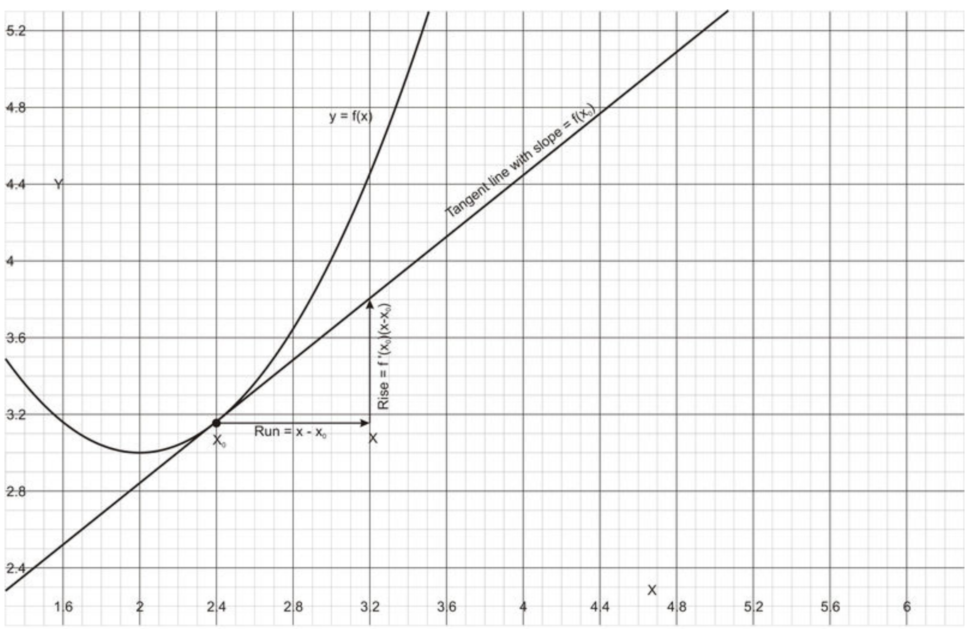


Figure 8b

So for values of x close to x_0, the values of y of this tangent line will closely approximate $f(x)$. This gives the approximation

$$f(x) = f(x_0) + f'(x_0)(x - x_0).$$

The Tangent Line Approximation (Linearization)

If f is a differentiable function at $x = x_0$, then the approximation function

$$L(x) = f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

is a linearization of f near x_0.

Example 1:

Find the linearization of $f(x) = \sqrt{x+3}$ at point $x = 1$.

Solution:

Taking the derivative of $f(x)$,

$$f'(x) = \frac{1}{2}(x+3)^{-1/2},$$

we have $f(1) = \sqrt{4} = 2, f'(1) = 1/4$, and

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &\approx 2 + \frac{1}{4}(x-1) \\ &\approx \frac{1}{4}x + \frac{7}{4}. \end{aligned}$$

This tells us that near the point $x = 1$, the function $f(x) = \sqrt{x+3}$ approximates the line $y = (x/4) + 7/4$. As we move away from $x = 1$, we lose accuracy (Figure 9).

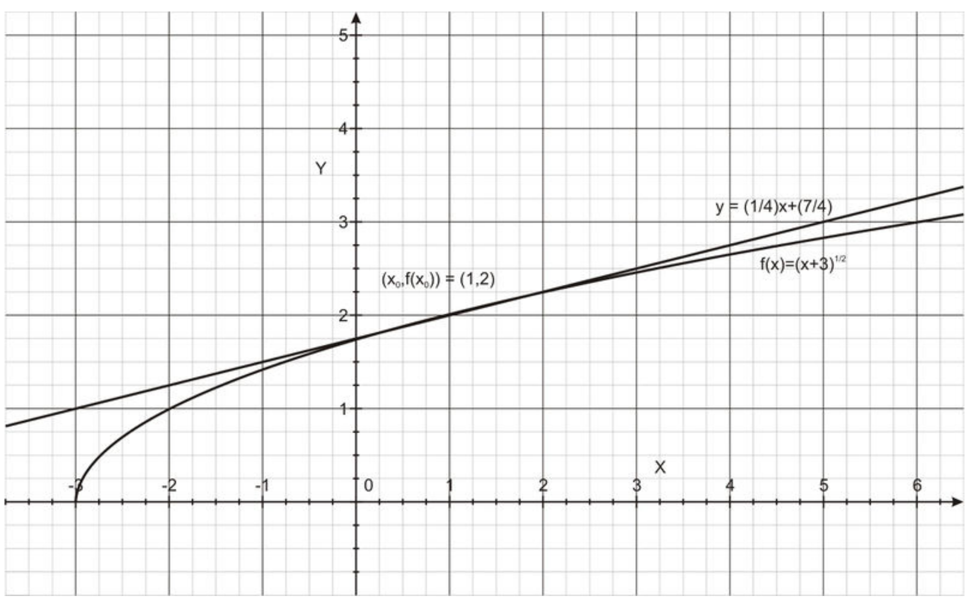


Figure 9

Example 2:

Find the linearization of $y = \sin x$ at $x = \pi/3$.

Solution:

Since $f(\pi/3) = \sin(\pi/3) = \sqrt{3}/2$, and $f'(x) = \cos x, f'(\pi/3) = \cos(\pi/3) = 1/2$, we have

$$\begin{aligned} f(x) &\approx \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) \\ &\approx \frac{\sqrt{3}}{2} + \frac{x}{2} - \frac{\pi}{6} \\ &\approx \frac{x}{2} + 0.343. \end{aligned}$$

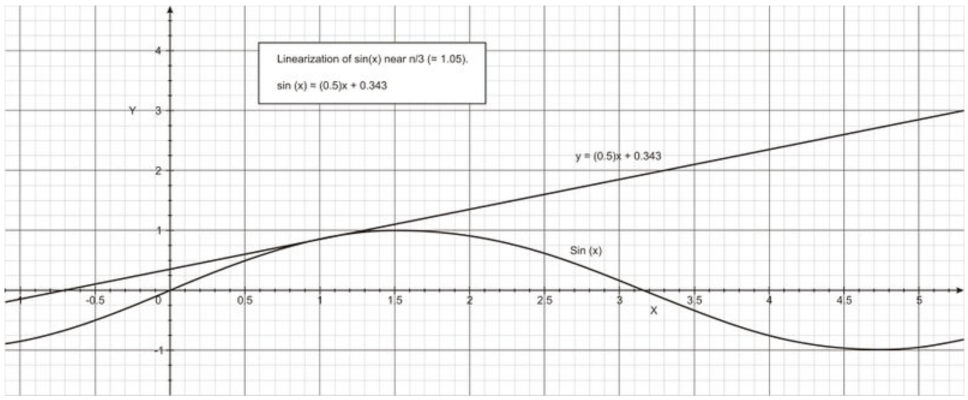


Figure 10

Newton's Method

When faced with a mathematical problem that cannot be solved with simple algebraic means, such as finding the roots of the polynomial $x^3 - 2x + 3 = 0$, calculus sometimes provides a way of finding the approximate solutions.

Let's say we are interested in computing $\sqrt{5}$ without using a calculator or a table. To do so, think about this problem in a different way. Assume that we are interested in solving the quadratic equation

$$f(x) = x^2 - 5 = 0$$

which leads to the roots $x = \pm\sqrt{5}$.

The idea here is to find the linearization of the above function, which is a straight-line equation, and then solve the linear equation for x.

Since

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$

or

$$2 < \sqrt{5} < 3,$$

We choose the linear approximation of $f(x)$ to be near $x_0 = 2$. Since $f(x) = x^2 - 5, f'(x) = 2x$ and thus $f(2) = -1$ and $f'(2) = 4$. Using the linear approximation formula,

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &\approx -1 + (4)(x-2) \\ &\approx -1 + 4x - 8 \\ &\approx 4x - 9. \end{aligned}$$

Notice that this equation is much easier to solve than $f(x) = x^2 - 5$. Setting $f(x) = 0$ and solving for x, we obtain,

$$\begin{aligned} 4x - 9 &= 0 \\ x &= \frac{9}{4} \\ &= 2.25. \end{aligned}$$

If you use a calculator, you will get $x = 2.236\ldots$ As you can see, this is a fairly good approximation. To be sure, calculate the *percent difference* [% diff] between the actual value and the approximate value,

$$\%\text{ diff} = \frac{2|A - X|}{|A + X|} 100\%,$$

where A is the accepted value and X is the calculated value.

$$\begin{aligned} \%\text{ diff} &= \frac{2|2.236 - 2.25|}{|2.236 + 2.25|} 100\% \\ &= 0.62\%, \end{aligned}$$

which is less than 1%.

We can actually make our approximation even better by repeating what we have just done not for $x = 2$, but for $x_1 = 2.25 = \frac{9}{4}$, a number that is even closer to the actual value of $\sqrt{5}$. Using the linear approximation again,

$$\begin{aligned} f(x) &\approx f(x_1) + f'(x_1)(x - x_1) \\ &\approx \frac{1}{16} + \frac{9}{2}\left(x - \frac{9}{4}\right) \\ &\approx \frac{9}{2}x - \frac{161}{16}. \end{aligned}$$

Solving for x by setting $f(x) = 0$, we obtain

$$x = x_2 = 2.236111,$$

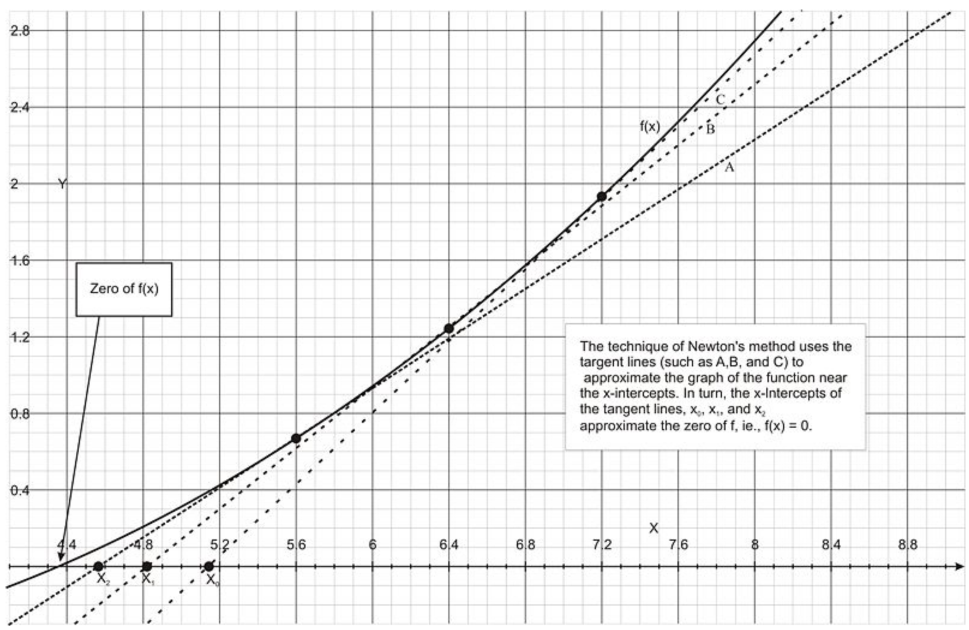
which is even a better approximation than $x_1 = 9/4$. We could continue this process generating a better approximation to $\sqrt{5}$. This is the basic idea of *Newton's Method*.

Here is a summary of Newton's method.

Newton's Method

1. Guess the first approximation to a solution of the equation $f(x) = 0$. A graph would be very helpful in finding the first approximation (see figure below).
2. Use the first approximation to find the second, the second to find the third and so on by using the recursion relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



Example 3:

Use Newton's method to find the roots of the polynomial $f(x) = x^3 + x - 1$.

Solution:

$$\begin{aligned} f(x) &= x^3 + x - 1 \\ f'(x) &= 3x^2 + 1. \end{aligned}$$

Using the recursion relation,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}. \end{aligned}$$

To help us find the first approximation, we make a graph of $f(x)$. As Figure 11 suggests, set $x_1 = 0.6$. Then using the recursion relation, we can generate $x_2, x_3, \ldots$.

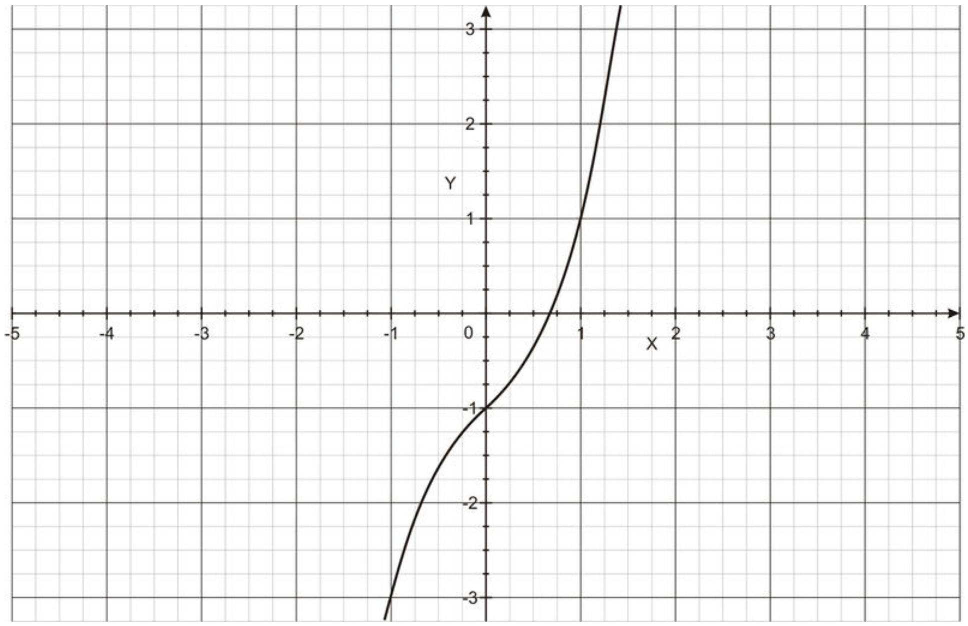


Figure 11

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} \\ x_2 &= 0.6 - \frac{(0.6)^3 + (0.6) - 1}{3(0.6)^2 + 1} \\ &= 0.6884615. \end{aligned}$$

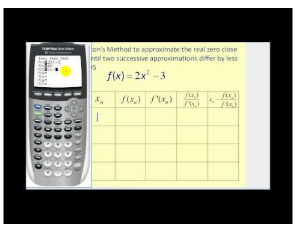
Using the recursion relation again to find x_3, we get

$$x_3 = 0.6836403.$$

We conclude that the solution to the equation $x^3 + x - 1 = 0$ is about 0.6836403.

Multimedia Links

For a video presentation of Newton's method **(10.0)**, see Math Video Tutorials by James Sousa, Newton's method (9:48).



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Review Questions

1. Find the linearization of

$$f(x) = \frac{x^2+1}{x}$$

at $a = 1$.
2. Find the linearization of $f(x) = \tan x$ at $a = \pi$.
3. Use the linearization method to show that when $x \ll 1$ (much less than 1), then $(1+x)^n \approx 1+nx$. Hint: Let $x = 0$.
4. Use the result of problem #3, $(1+x)^n \approx 1+nx$, to find the approximation for the following:
   a. $f(x) = (1-x)^4$
   b. $f(x) = \sqrt{1-x}$
   c. $f(x) = \frac{5}{\sqrt{1+x}}$
   d. Without using a calculator, approximate $(1.003)^{99}$.
5. Use Newton's Method to find the roots of $x^3+3 = 0$.
6. Use Newton's Method to find the roots of $-x+3\sqrt{-1+x} = 0$.

$$f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = n(u)^{n-1} \cdot (1) = n(1+x)^{n-1}$$

If $x << 1$, we can use $x_0 = 0$ and linearize around the point $f(x_0) = f(0)$:

$$\begin{aligned}
y &= f(x_0) + f'(x_0)(x-x_0) \\
y &= (1+0)^n + n(1+0)^{n-1}(x-0) \\
y &= (1)^n + n(1)^{n-1}(x) \\
y &= 1+nx
\end{aligned}$$

4.
   a. $1-4x$
   b. $1-\frac{1}{2}x$
   c. $5-\frac{5}{2}x$
   d. 1.297
5. $x \approx -1.442$
6. $x \approx 1.146$ and $x \approx 7.854$

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9727 .

Chapter 3 Applications of Derivatives

Chapter Outline

3.1 Related Rates

Learning Objectives

A student will be able to:

- Solve problems that involve related rates.

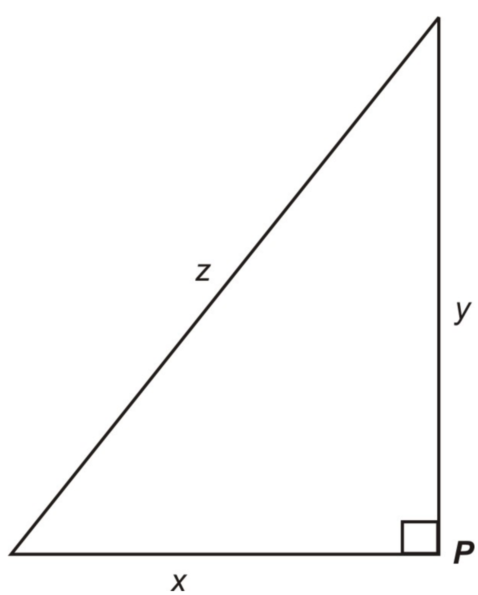
Introduction

In this lesson we will discuss how to solve problems that involve related rates. Related rate problems involve equations where there is some relationship between two or more derivatives. We solved examples of such equations when we studied implicit differentiation in Lesson 2.6. In this lesson we will discuss some real-life applications of these equations and illustrate the strategies one uses for solving such problems.

Let's start our discussion with some familiar geometric relationships.

Example 1: ***Pythagorean Theorem***

$$x^2 + y^2 = z^2$$



We could easily attach some real-life situation to this geometric figure. Say for instance that x and y represent the paths of two people starting at point p and walking North and West, respectively, for two hours. The quantity z represents the distance between them at any time t. Let's now see some relationships between the various rates of change that we get by implicitly differentiating the original equation $x^2 + y^2 = z^2$ with respect to time t.

$$x^2 + y^2 = z^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

Simplifying, we have

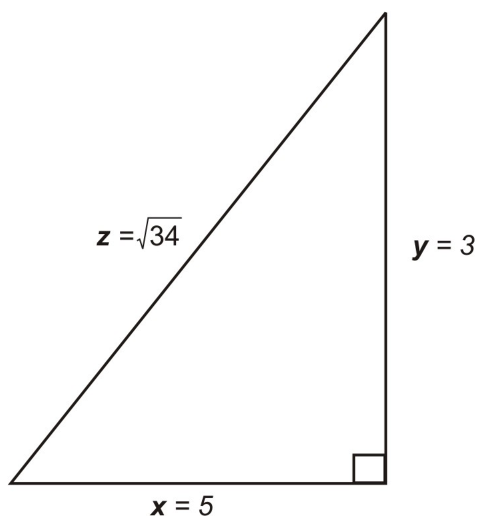
Equation 1. $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$

So we have relationships between the derivatives, and since the derivatives are rates, this is an example of ***related rates***. Let's say that person x is walking at 5 mph and that person y is walking at 3 mph. The rate at which the distance between the two walkers is changing at any time is dependent on the rates at which the two people are walking. Can you think of any problems you could pose based on this information?

One problem that we could pose is at what rate is the distance between x and y increasing after one hour. That is, find dz/dt.

Solution:

Assume that they have walked for one hour. So $x = 5$ mi and $y = 3$. Using the Pythagorean Theorem, we find the distance between them after one hour is $z = \sqrt{34} = 5.83$ miles.



If we substitute these values into **Equation 1** along with the individual rates we get

$$5(5) + 3(3) = \sqrt{34}\frac{dz}{dt}$$
$$34 = \sqrt{34}\frac{dz}{dt}$$
$$\frac{34}{\sqrt{34}} = \frac{dz}{dt}.$$

Hence after one hour the distance between the two people is increasing at a rate of $\frac{dz}{dt} = \frac{34}{\sqrt{34}} \approx 5.83$ mph.

Our second example lists various formulas that are found in geometry.

As with the Pythagorean Theorem, we know of other formulas that relate various quantities associated with geometric shapes. These present opportunities to pose and solve some interesting problems

Example 2: ***Perimeter and Area of a Rectangle***

We are familiar with the formulas for Perimeter and Area:

$$P = 2 * l + 2 * w,$$
$$A = l * w.$$

Suppose we know that at an instant of time, the length is changing at the rate of 8 ft/hour and the perimeter is changing at a rate of 24 ft/hour. At what rate is the width changing at that instant?

Solution:

If we differentiate the original equation, we have

Equation 2: $\frac{dp}{dt} = 2 * \frac{dl}{dt} + 2 * \frac{dw}{dt}$.

Substituting our known information into Equation II, we have

$$24 = (2 * 8) + 2 * \frac{dw}{dt}$$
$$8 = 2 * \frac{dw}{dt}$$
$$4 = \frac{dw}{dt}.$$

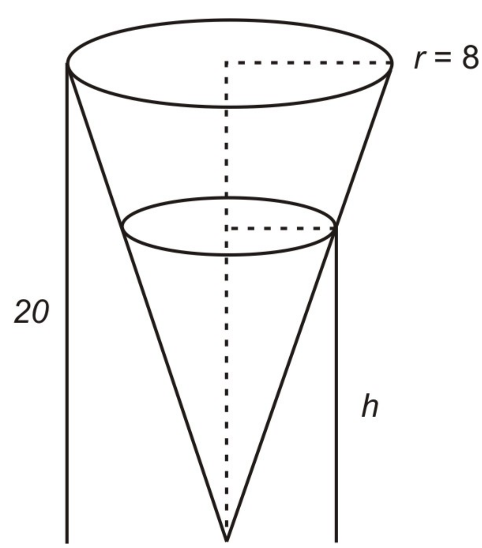
The width is changing at a rate of 4 ft/hour.

Okay, rather than providing a related rates problem involving the area of a rectangle, we will leave it to you to make up and solve such a problem as part of the homework (HW #1).

Let's look at one more geometric measurement formula.

Example 3: ***Volume of a Right Circular Cone***

$$V = \frac{1}{3}\pi r^2 h$$



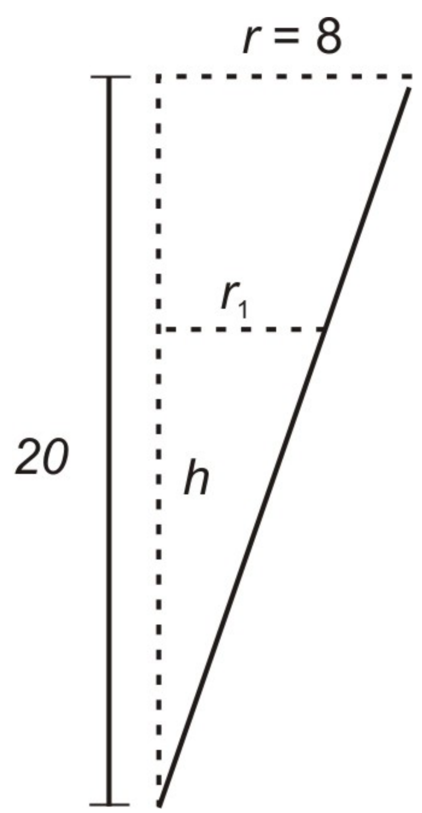
We have a water tank shaped as an inverted right circular cone. Suppose that water flows into the tank at the rate of 5 ft^3/min. At what rate is the water level rising when the height of the water in the tank is 6 feet?

Solution:

We first note that this problem presents some challenges that the other examples did not. Both r and h are functions of t, and so implicit differentiation of the $V(t)$ function is going to produce several variables in the form $r(t)$, $\frac{dr}{dt}$, $h(t)$, and $\frac{dh}{dt}$, and that will give us too many variables to solve for $\frac{dh}{dt}$. So we need to find a way to eliminate r and $\frac{dr}{dt}$. When we differentiate the original equation, $V = (1/3)\pi r^2 h$, we get

$$\frac{dV}{dt} = \frac{1}{3}\pi(h)(2r)\frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}.$$

The difficulty here is that we have no information about the radius when the water level is at 6 feet. So we need to relate the radius a quantity that we do know something about. Starting with the original equation, let's find a relationship between h and r. Let r_1 be the radius of the surface of the water as it flows out of the tank.



Note that the two triangles are similar and thus corresponding parts are proportional. In particular,

$$\frac{r_1}{h} = \frac{8}{20}$$
$$r_1 = \frac{8h}{20} = \frac{2h}{5}.$$

Now we can solve the problem by substituting $r_1 = (2h/5)$ into the original equation:

$$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3.$$

Hence $\frac{dV}{dt} = \frac{12\pi}{75}h^2\frac{dh}{dt}$, and by substitution,

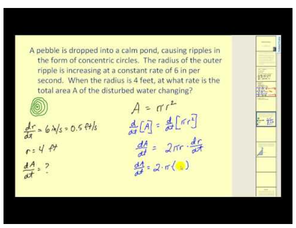
$$5 = \frac{12\pi}{75}(36)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{375}{432\pi} \approx 0.28\frac{\text{ft}}{\text{min}}.$$

Lesson Summary

1. We learned to solve problems that involved related rates.

Multimedia Links

For a video presentation of related rates **(12.0)**, see Math Video Tutorials by James Sousa, Related Rates (10:34).



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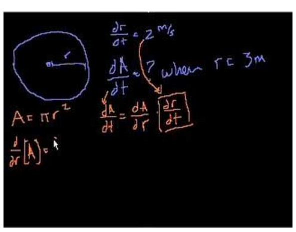
In the following applet you can explore a problem about a melting snowball where the radius is decreasing at a constant rate. Calculus Applets Snowball Problem . Experiment with changing the time to see how the volume *does not* change at a constant rate in this problem. If you'd like to see a video of another example of a related rate problem worked out **(12.0)**, see Khan Academy Rates of Change (Part 2) (5:37).



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Review Questions

1.
   a. Make up a related rates problem about the area of a rectangle.
   b. Illustrate the solution to your problem.
2. Suppose that a particle is moving along the curve $4x^2 + 16y^2 = 32$. When it reaches the point $(2,1)$, the $x-$coordinate is increasing at a rate of 3 ft/sec. At what rate is the $y-$coordinate changing at that instant?

3. A regulation softball diamond is a square with each side of length 60 ft. Suppose a player is running from first base to second base at a speed of 18 ft/sec. At what rate is the distance between the runner and home plate changing when the runner is $2/3$ of the way from first to second base?
4. At a recent Hot Air Balloon festival, a hot air balloon was released. Upon reaching a height of 300 ft, it was rising at a rate of 20 ft/sec. Mr. Smith was 100 ft away from the launch site watching the balloon. At what rate was the distance between Mr. Smith and the balloon changing at that instant?
5. Two trains left the St. Louis train station in the late morning. The first train was traveling East at a constant speed of 65 mph. The second train traveled South at a constant speed of 75 mph. At 3 PM, the first train had traveled a distance of 120 miles while the second train had traveled a distance of 130 miles. How fast was the distance between the two trains changing at that time?
6. Suppose that a 17 ft ladder is sliding down a wall at a rate of -6 ft/sec. At what rate is the distance between the bottom of the ladder and the wall increasing when the top is 8 ft from the ground?
7. Suppose that the length of a rectangle is increasing at the rate of 6 ft/min and the width is increasing at a rate of 2 ft/min. At what rate is the area of the rectangle changing when its length is 25 ft and its width is 15 ft?
8. Suppose that the quantity demand of new $40''$ plasma TVs is related to its unit price by the formula $p+x^2 = 1200$, where p is measured in dollars and x is measured in units of one thousand. How is the quantity demand changing when $x = 20, p = 1500$, and the price per TV is decreasing at a rate of \$10 per week?
9. The volume of a cube with side s is changing. At a certain instant, the sides of the cube are 6 inches and increasing at the rate of $1/4$ in/min. How fast is the volume of the cube increasing at that time?
10.
    a. Suppose that the area of a circle is increasing at a rate of 24 $\text{in}^2/$ min. How fast is the radius increasing when the area is $36\pi\ \text{in}^2$?
    b. How fast is the circumference changing at that instant?

3.2 Extrema and the Mean Value Theorem

Learning Objectives

A student will be able to:

- Solve problems that involve extrema.
- Study Rolle's Theorem.
- Use the Mean Value Theorem to solve problems.

Introduction

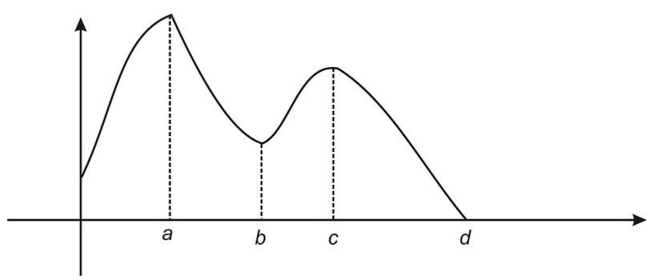
In this lesson we will discuss a second application of derivatives, as a means to study extreme (maximum and minimum) values of functions. We will learn how the maximum and minimum values of functions relate to derivatives.

Let's start our discussion with some formal working definitions of the maximum and minimum values of a function.

Definition

A function f has a **maximum** at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f. Similarly, f has a ***minimum*** at $x = a$ if $f(a) \leq f(x)$ for all x in the domain of f. The values of the function for these $x-$values are called ***extreme*** values or ***extrema.***

Here is an example of a function that has a maximum at $x = a$ and a minimum at $x = d$:



Observe the graph at $x = b$. While we do not have a minimum at $x = b$, we note that $f(b) \leq f(x)$ for all x near b. We say that the function has a **local minimum** at $x = b$. Similarly, we say that the function has a ***local maximum*** at $x = c$ since $f(c) \geq f(x)$ for some x contained in open intervals of c.

Let's recall the Min-Max Theorem that we discussed in lesson on Continuity.

Min-Max Theorem: If a function $f(x)$ is continuous in a closed interval I, then $f(x)$ has both a maximum value and a minimum value in I. In order to understand the proof for the Min-Max Theorem conceptually, attempt to draw a function on a closed interval (including the endpoints) so that no point is at the highest part of the graph. No matter how the function is sketched, there will be at least one point that is highest.

We can now relate extreme values to derivatives in the following Theorem by the French mathematician Fermat.

Theorem: If $f(c)$ is an extreme value of f for some open interval of c, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: The theorem states that if we have a local max or local min, and if $f'(c)$ exists, then we must have $f'(c) = 0$.

Suppose that f has a local max at $x = c$. Then we have $f(c) \geq f(x)$ for some open interval $(c-h, c+h)$ with $h > 0$.

So $f(c+h) - f(c) \leq 0$.

Consider $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$.

Since $f(c+h) - f(c) \leq 0$, we have $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq \lim_{h\to 0^+} 0 = 0$.

Since $f'(c)$ exists, we have $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$, and so $f'(c) \leq 0$.

If we take the left-hand limit, we get $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$.

Hence $f'(c) \geq 0$ and $f'(c) \leq 0$ it must be that $f'(c) = 0$.

If $x = c$ is a local minimum, the same argument follows.

Definition

We will call $x = c$ a ***critical value*** in $[a,b]$ if $f'(c) = 0$ or $f'(c)$ does not exist, or if $x = c$ is an endpoint of the interval.

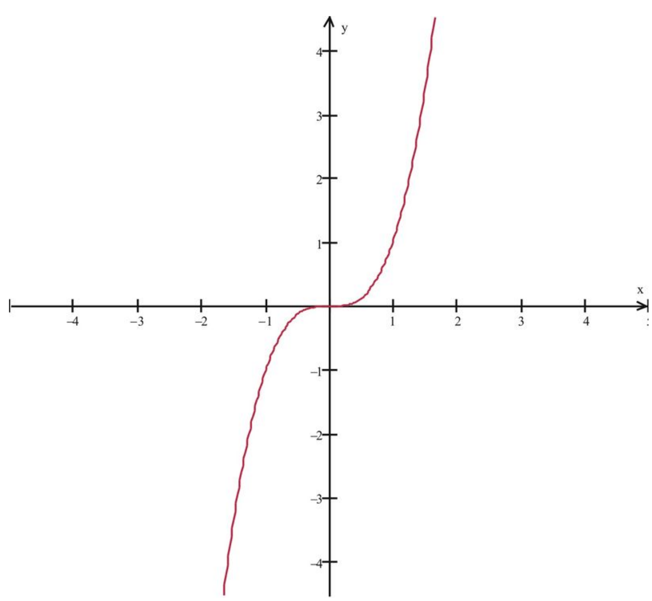
We can now state the Extreme Value Theorem.

Extreme Value Theorem: If a function $f(x)$ is continuous in a closed interval $[a,b]$, with the maximum of f at $x = c_1$ and the minimum of f at $x = c_2$, then c_1 and c_2 are critical values of f.

Proof: The proof follows from Fermat's theorem and is left as an exercise for the student.

Example 1:

Let's observe that the converse of the last theorem is not necessarily true: If we consider $f(x) = x^3$ and its graph, then we see that while $f'(0) = 0$ at $x = 0$,$x = 0$ is not an extreme point of the function.

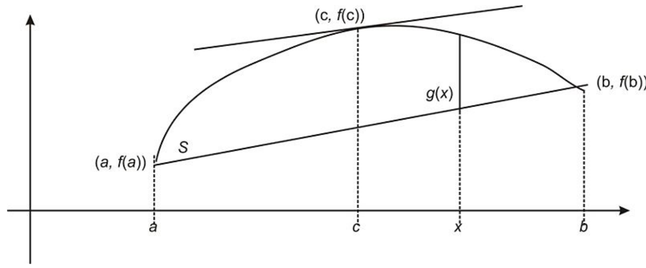


Rolle's Theorem: If f is continuous and differentiable on a closed interval $[a,b]$ and if $f(a) = f(b)$, then f has at least one value c in the open interval (a,b) such that $f'(c) = 0$.

The proof of Rolle's Theorem can be found at http://en.wikipedia.org/wiki/Rolle's_theorem .

Mean Value Theorem: If f is a continuous function on a closed interval $[a,b]$ and if f' contains the open interval (a,b) in its domain, then there exists a number c in the interval (a,b) such that $f(b) - f(a) = (b-a)f'(c)$.

Proof: Consider the graph of f and secant line s as indicated in the figure.



By the Point-Slope form of line s we have

$y - f(a) = m(x-a)$ and $y = m(x-a) + f(a)$.

For each x in the interval (a,b), let $g(x)$ be the vertical distance from line S to the graph of f. Then we have

$g(x) = f(x) - [m(x-a) + f(a)]$ for every x in (a,b).

Note that $g(a) = g(b) = 0$. Since g is continuous in $[a,b]$ and g' exists in (a,b), then Rolle's Theorem applies. Hence there exists c in (a,b) with $g'(c) = 0$.

So $g'(x) = f'(x) - m$ for every x in (a,b).

In particular,

$g'(c) = f'(c) - m = 0$ and

$$\begin{aligned} f'(c) &= m \\ f'(c) &= \frac{f(b)-f(a)}{b-a} \\ f(b)-f(a) &= (b-a)f'(c). \end{aligned}$$

The proof is complete.

Example 2:

Verify that the Mean Value Theorem applies for the function $f(x) = x^3 + 3x^2 - 24x$ on the interval $[1,4]$.

Solution:

We need to find c in the interval $(1,4)$ such that $f(4) - f(1) = (4-1)f'(c)$.

Note that $f'(x) = 3x^2 + 6x - 24$, and $f(4) = 16, f(1) = -20$. Hence, we must solve the following equation:

$$\begin{aligned} 36 &= 3f'(c) \\ 12 &= f'(c). \end{aligned}$$

By substitution, we have

$$\begin{aligned} 12 &= 3c^2 + 6c - 24 \\ 3c^2 + 6c - 36 &= 0 \\ c^2 + 2c - 12 &= 0 \\ c &= \frac{-2 \pm \sqrt{52}}{2} \approx -4.61, 2.61. \end{aligned}$$

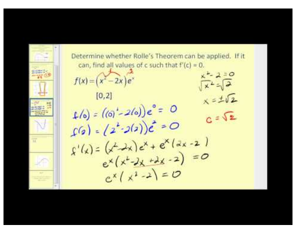
Since we need to have c in the interval $(1,4)$, the positive root is the solution, $c = \frac{-2+\sqrt{52}}{2} \approx 2.61$.

Lesson Summary

1. We learned to solve problems that involve extrema.
2. We learned about Rolle's Theorem.
3. We used the Mean Value Theorem to solve problems.

Multimedia Links

For a video presentation of Rolle's Theorem **(8.0)**, see Math Video Tutorials by James Sousa, Rolle's Theorem (7:54).



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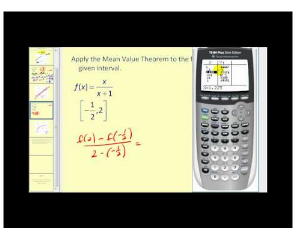
For more information about the Mean Value Theorem **(8.0)**, see Math Video Tutorials by James Sousa, Mean Value Theorem (9:52).



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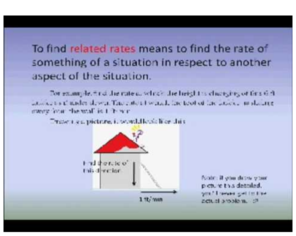
For a well-done, but unorthodox, student presentation of the Extreme Value Theorem and Related Rates **(3.0)(12.0)**, see Extreme Value Theorem (10:00).



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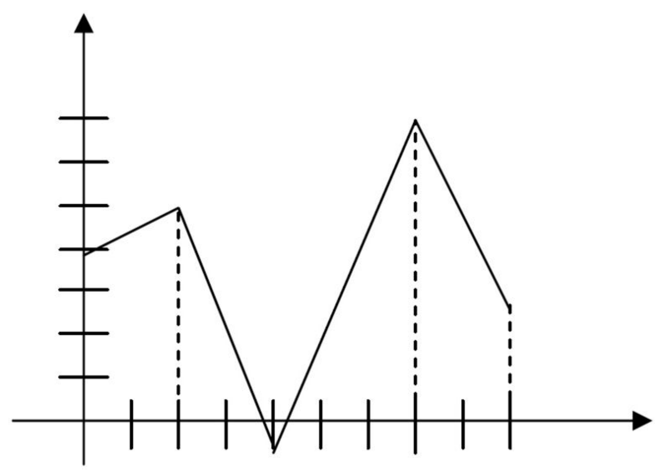
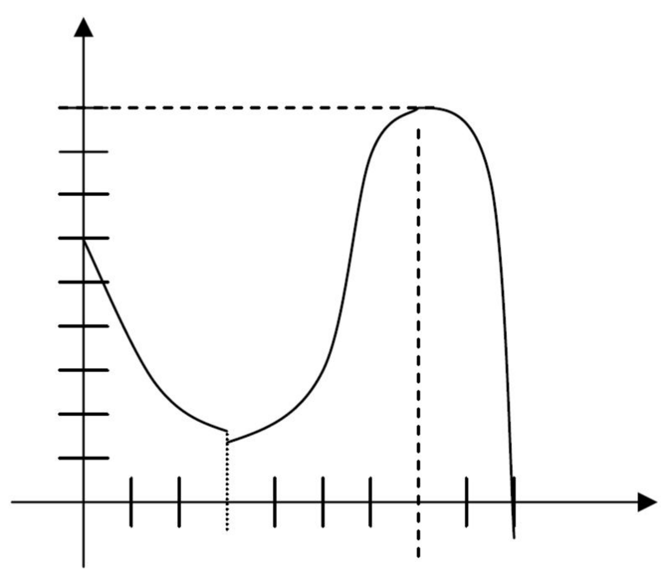
Review Questions

In problems #1–3, identify the absolute and local minimum and maximum values of the function (if they exist); find the extrema. (Units on the axes indicate 1 unit).

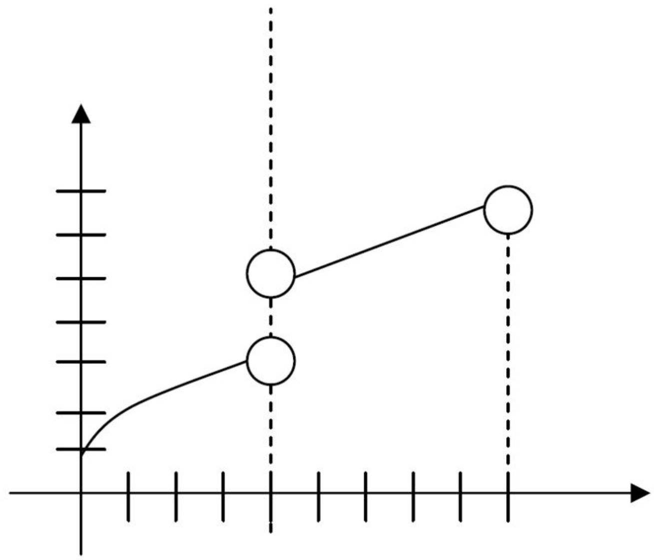
1. Continuous on $[0,9]$



2. Continuous on $[0,3]\cup(3,9]$



3. Continuous on $[0,4]\cup(4,9)$



In problems #4–6, find the extrema and sketch the graph.

4. $f(x) = -x^2 - 6x + 4, [-4, 1]$
5. $f(x) = x^3 - x^4, [0, 2]$
6. $f(x) = -x^2 + \frac{4}{x^2}, [-2, 0]$
7. Verify Rolle's Theorem by finding values of x for which $f(x) = 0$ and $f'(x) = 0. f(x) = 3x^3 - 12x$
8. Verify Rolle's Theorem for $f(x) = x^2 - \frac{2}{x-1}$.
9. Verify that the Mean Value Theorem works for $f(x) = \frac{(x+2)}{x}, [1, 2]$.
10. Prove that the equation $x^3 + a_1x^2 + a_2x = 0$ has a positive root at $x = r$, and that the equation $3x^2 + 2a_1x + a_2 = 0$ has a positive root less than r.

3.3 The First Derivative Test

Learning Objectives

A student will be able to:

- Find intervals where a function is increasing and decreasing.
- Apply the First Derivative Test to find extrema and sketch graphs.

Introduction

In this lesson we will discuss increasing and decreasing properties of functions, and introduce a method with which to study these phenomena, the First Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph. Note on notation: The symbol ε and $\in$ are equivalent and denote that a particular element is contained within a particular set.

Definition

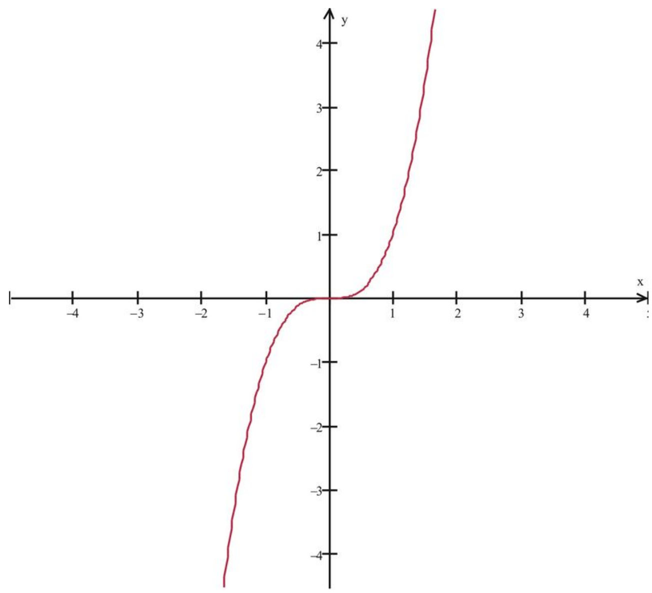
A function f is said to be ***increasing*** on $[a,b]$ contained in the domain of f if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$ for all $x_1, x_2 \in [a,b]$. A function f is said to be ***decreasing*** on $[a,b]$ contained in the domain of f if $f(x_1) \geq f(x_2)$ whenever $x_1 \geq x_2$ for all $x_1, x_2 \in [a,b]$.

If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all $x_1, x_2 \in [a,b]$, then we say that f is ***strictly increasing*** on $[a,b]$. If $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ for all $x_1, x_2 \in [a,b]$, then we say that f is ***strictly decreasing*** on $[a,b]$.

We saw several examples in the Lesson on Extreme and the Mean Value Theorem of functions that had these properties.

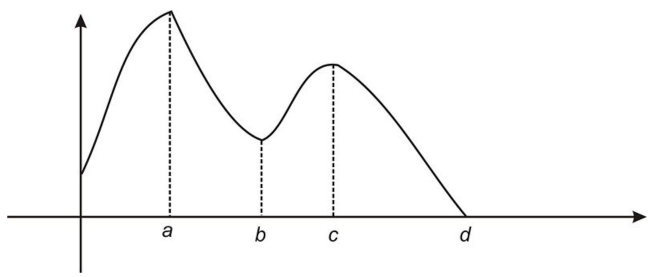
Example 1:

The function $f(x) = x^3$ is strictly increasing on $(-\infty, +\infty)$:



Example 2:

The function indicated here is strictly increasing on $(0,a)$ and (b,c), and strictly decreasing on (a,b) and (c,d).



We can now state the theorems that relate derivatives of functions to the increasing/decreasing properties of functions.

Theorem: If f is continuous on interval $[a,b]$, then:

1. If $f'(x) > 0$ for every $x \in [a,b]$, then f is strictly increasing in $[a,b]$.
2. If $f'(x) < 0$ for every $x \in [a,b]$, then f is strictly decreasing in $[a,b]$.

Proof: We will prove the first statement. A similar method can be used to prove the second statement and is left as an exercise to the student.

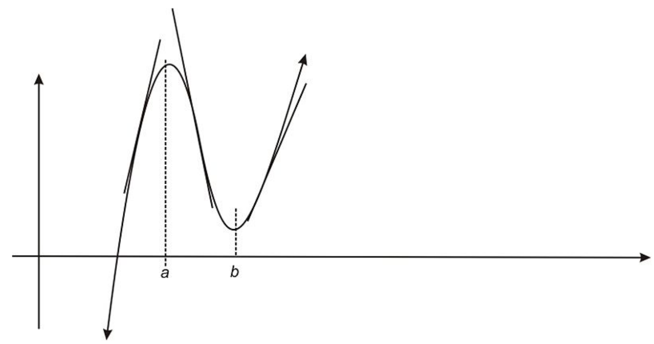
Consider $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c).$$

By assumption, $f'(x) > 0$ for every $x \in [a,b]$; hence $f'(c) > 0$. Also, note that $x_2 - x_1 > 0$.

Hence $f(x_2) - f(x_1) > 0$ and $f(x_2) > f(x_1)$.

We can observe the consequences of this theorem by observing the tangent lines of the following graph. Note the tangent lines to the graph, one in each of the intervals $(0,a)$, (a,b),$(b,+\infty)$.



Note first that we have a relative maximum at $x = a$ and a relative minimum at $x = b$. The slopes of the tangent lines change from positive for $x \in (0,a)$ to negative for $x \in (a,b)$ and then back to positive for $x \in (b,+\infty)$. From this we example infer the following theorem:

First Derivative Test

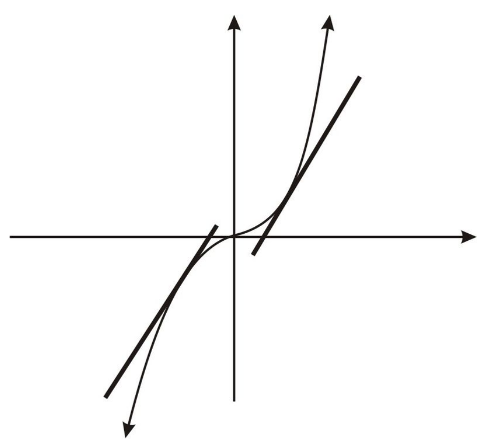
Suppose that f is a continuous function and that $x = c$ is a critical value of f. Then:

1. If f' changes from positive to negative at $x = c$, then f has a local maximum at $x = c$.
2. If f' changes from negative to positive at $x = c$, then f has a local minimum at $x = c$.
3. If f' does not change sign at $x = c$, then f has neither a local maximum nor minimum at $x = c$.

Proof of these three conclusions is left to the reader.

Example 3:

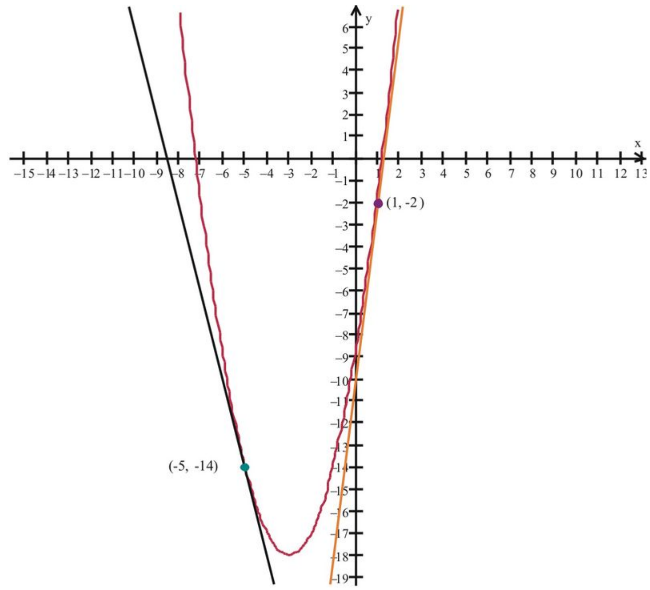
Our previous example showed a graph that had both a local maximum and minimum. Let's reconsider $f(x) = x^3$ and observe the graph around $x = 0$. What happens to the first derivative near this value?



We observe that the tangent lines to the graph are positive on both sides of $x = 0$. The first derivative test ($f'(x) = 3x^2$) verifies this fact, and that the slopes of the tangent line are positive for all nonzero x. Although $f'(0) = 0$, and so f has a critical value at $x = 0$, the third part of the First Derivative Test tells us that the failure of f' to change sign at $x = 0$ means that f has neither a local minimum nor a local maximum at $x = 0$.

Example 4:

Let's consider the function $f(x) = x^2 + 6x - 9$ and observe the graph around $x = -3$. What happens to the first derivative near this value?



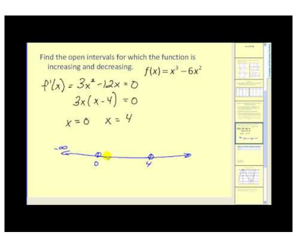
We observe that the slopes of the tangent lines to the graph change from negative to positive at $x = -3$. The first derivative test verifies this fact. Note that the slopes of the tangent lines to the graph are negative for $x \in (-\infty, -3)$ and positive for $x \in (-3, -\infty)$.

Lesson Summary

1. We found intervals where a function is increasing and decreasing.
2. We applied the First Derivative Test to find extrema and sketch graphs.

Multimedia Links

For more examples on determining whether a function is increasing or decreasing **(9.0)**, see Math Video Tutorials by James Sousa, Determining where a function is increasing and decreasing using the first derivative (10:05).



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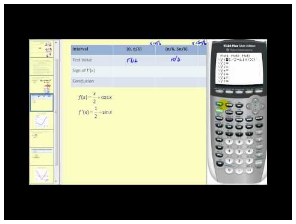
For a video presentation of increasing and decreasing trigonometric functions and relative extrema **(9.0)**, see Math Video Tutorials by James Sousa, Increasing and decreasing trig functions, relative extrema (6:03).



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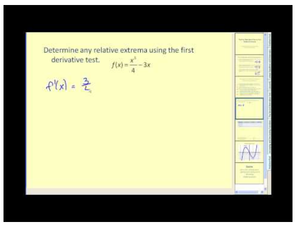
For more information on finding relative extrema using the first derivative **(9.0)**, see Math Video Tutorials by James Sousa, Finding relative extrema using the first derivative (6:18).



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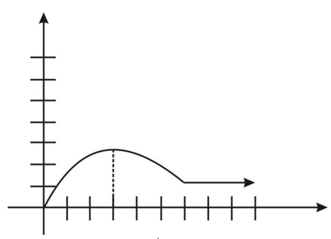
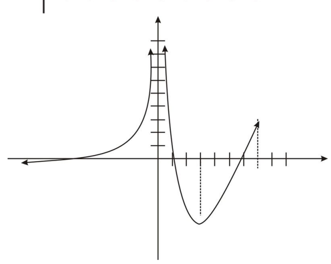
Review Questions

In problems #1–2, identify the intervals where the function is increasing, decreasing, or is constant. (Units on the axes indicate single units).

1. 
2. 
3. Give the sign of the following quantities for the graph in #2.
   a. $f'(-3)$
   b. $f'(1)$
   c. $f'(3)$
   d. $f'(4)$

For problems #4–6, determine the intervals in which the function is increasing and those in which it is decreasing. Sketch the graph.

4. $f(x) = x^2 - \frac{1}{x}$
5. $f(x) = (x^2 - 1)^5$
6. $f(x) = (x^2 - 1)^4$

For problems #7–10:

a. Use the First Derivative Test to find the intervals where the function increases and/or decreases b. Identify all max, mins, or relative max and mins c. Sketch the graph

7. $f(x) = -x^2 - 4x - 1$
8. $f(x) = x^3 + 3x^2 - 9x + 1$
9. $f(x) = x^{\frac{2}{3}}(x - 5)$
10. $f(x) = 2x\sqrt{x^2 + 1}$

3.4 The Second Derivative Test

Learning Objectives

A student will be able to:

- Find intervals where a function is concave upward or downward.
- Apply the Second Derivative Test to determine concavity and sketch graphs.

Introduction

In this lesson we will discuss a property about the shapes of graphs called concavity, and introduce a method with which to study this phenomenon, the Second Derivative Test. This method will enable us to identify precisely the intervals where a function is either increasing or decreasing, and also help us to sketch the graph.

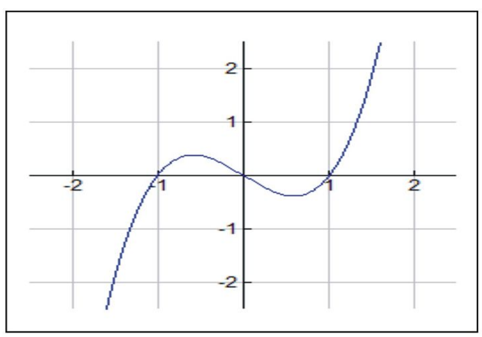
Definition

A function f is said to be ***concave upward*** on $[a,b]$ contained in the domain of f if f' is an increasing function on $[a,b]$ and ***concave downward*** on $[a,b]$ if f' is a decreasing function on $[a,b]$.

Here is an example that illustrates these properties.

Example 1:

Consider the function $f(x) = x^3 - x$:



The function has zeros at $x = \pm 1, 0$ and has a relative maximum at $x = -\frac{\sqrt{3}}{3}$ and a relative minimum at $x = \frac{\sqrt{3}}{3}$. Note that the graph appears to be concave down for all intervals in $(-\infty, 0)$ and concave up for all intervals in $(0, +\infty)$. Where do you think the concavity of the graph changed from concave down to concave up? If you answered at $x = 0$ you would be correct. In general, we wish to identify both the extrema of a function and also the points where the graph changes concavity. The following definition provides a formal characterization of such points.

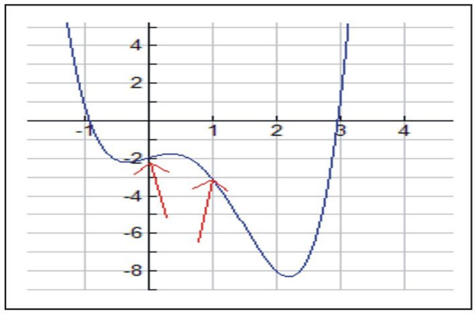
Definition

A point on a graph of a function f where the concavity changes is called an ***inflection point***.

The example above had only one inflection point. But we can easily come up with examples of functions where there is more than one point of inflection.

Example 2:

Consider the function $f(x) = x^4 - 3x^3 + x - 2$.



We can see that the graph has two relative minimums, one relative maximum, and two inflection points (as indicated by arrows).

In general we can use the following two tests for concavity and determining where we have relative maximums, minimums, and inflection points.

Test for Concavity

Suppose that I is some interval $[a,b]$ in the domain of f and that f is continuous on I.

1. If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

A consequence of this concavity test is the following test to identify extreme values of f.

Second Derivative Test for Extrema

Suppose that f is a continuous function near c and that c is a critical value of f. Then

1. If $f''(c) > 0$, then f has a relative minimum at $x = c$.
2. If $f''(c) < 0$, then f has a relative maximum at $x = c$.
3. If $f''(c) = 0$, then the test is inconclusive and $x = c$ may be a point of inflection.

Recall the graph $f(x) = x^3$. We observed that $x = 0$, and that there was neither a maximum nor minimum. The Second Derivative Test cautions us that this may be the case since at $f''(0) = 0$ at $x = 0$.

So now we wish to use all that we have learned from the First and Second Derivative Tests to sketch graphs of functions. The following table provides a summary of the tests and can be a useful guide in sketching graphs.

TABLE 3.1:

Signs of first and second derivatives	Information from applying First and Second Derivative Tests	Shape of the graphs
$f'(x) > 0$ $f''(x) > 0$	f is increasing f is concave upward	
$f'(x) > 0$ $f''(x) < 0$	f is increasing f is concave downward	
$f'(x) < 0$ $f''(x) > 0$	f is decreasing f is concave upward	
$f'(x) < 0$ $f''(x) < 0$	f is decreasing f is concave downward	

Lets' look at an example where we can use both the First and Second Derivative Tests to find out information that will enable us to sketch the graph.

Example 3:

Let's examine the function $f(x) = x^5 - 5x + 2$.

1. Find the critical values for which $f'(c) = 0$.

$f'(x) = 5x^4 - 5 = 0$, or

$x^4 - 1 = 0$ at $x = \pm 1$.

Note that $f''(x) = 20x^3 = 0$ when $x = 0$.

2. Apply the First and Second Derivative Tests to determine extrema and points of inflection.

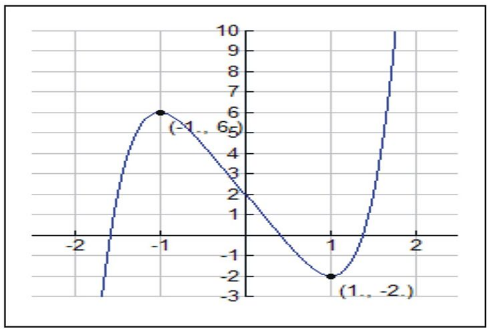
We can note the signs of f' and f'' in the intervals partioned by $x = \pm 1, 0$.

TABLE 3.2:

Key intervals	$f'(x)$	$f''(x)$	*Shape of graph*
$x < -1$	$+$	$-$	Increasing, concave down
$-1 < \times < 0$	$-$	$-$	Decreasing, concave down
$0 < \times < 1$	$-$	$+$	Decreasing, concave up
$x > 1$	$+$	$+$	Increasing, concave up

Also note that $f''(-1) = -20 < 0$. By the Second Derivative Test we have a relative maximum at $x = -1$, or the point $(-1, 6)$.

In addition, $f''(1) = 20 > 0$. By the Second Derivative Test we have a relative minimum at $x = 1$, or the point $(1, -2)$. Now we can sketch the graph.

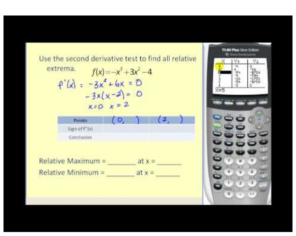


Lesson Summary

1. We learned to identify intervals where a function is concave upward or downward.
2. We applied the First and Second Derivative Tests to determine concavity and sketch graphs.

Multimedia Links

For a video presentation of the second derivative test to determine relative extrema **(9.0)**, see Math Video Tutorials by James Sousa, The Second Derivative Test (8:41).



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Review Questions

1. Find all extrema using the Second Derivative Test. $f(x) = \frac{x^2}{4} + \frac{4}{x}$
2. Consider $f(x) = x^2 + ax + b$, with $f(1) = 3$.
    a. Determine a and b so that $x = 1$ is a critical value of the function f.
    b. Is the point $(1,3)$ a maximum, a minimum or neither?

In problems #3–6, find all extrema and inflection points. Sketch the graph.

3. $f(x) = x^3 + x^2$
4. $f(x) = \frac{x^2+3}{x}$
5. $f(x) = x^3 - 12x$
6. $f(x) = -\frac{1}{4}x^4 + 2x^2$
7. Use your graphing calculator to examine the graph of $f(x) = x(x-1)^3$ (Hint: you will need to change the y range in the viewing window)
    a. Discuss the concavity of the graph in the interval $(0, \frac{1}{2})$.
    b. Use your calculator to find the minimum value of the function in the interval.
8. True or False: $f(x) = x^4 + 4x^3$ has a relative minimum at $x = -2$ and a relative maximum at $x = 0$?
9. If possible, provide an example of a non-polynomial function that has exactly one relative minimum.
10. If possible, provide an example of a non-polynomial function that is concave downward everywhere in its domain.

3.5 Limits at Infinity

Learning Objectives

A student will be able to:

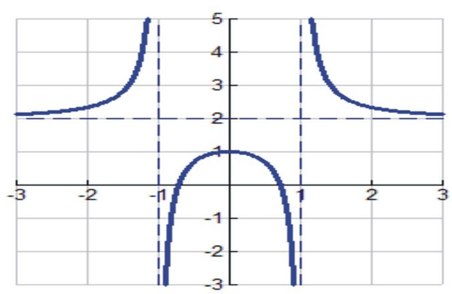
- Examine end behavior of functions on infinite intervals.
- Determine horizontal asymptotes.
- Examine indeterminate forms of limits of rational functions.
- Apply L'Hospital's Rule to find limits.
- Examine infinite limits at infinity.

Introduction

In this lesson we will return to the topics of infinite limits and end behavior of functions and introduce a new method that we can use to determine limits that have indeterminate forms.

Examine End Behavior of Functions on Infinite Intervals

Suppose we are trying to analyze the end behavior of rational functions. Let's say we looked at some rational functions such as $f(x) = \frac{2x^2-1}{x^2-1}$ and showed that $\lim_{x\to+\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$. We required an analysis of the end behavior of f since computing the limit by direct substitution yielded the indeterminate form $\frac{\infty}{\infty}$. Our approach to compute the infinite limit was to look at actual values of the function $f(x)$ as x approached $\pm\infty$. We interpreted the result graphically as the function having a horizontal asymptote at $f(x) = 2$.



We were then able to find infinite limits of more complicated rational functions such as $\lim_{x\to\infty} \frac{3x^4-2x^2+3x+1}{2x^4-2x^2+x-3} = \frac{3}{2}$ using the fact that $\lim_{x\to\infty} \frac{1}{x^p} = 0, p > 0$. Similarly, we used such an approach to compute limits whenever direct substitution resulted in the indeterminate form $\frac{0}{0}$, such as $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$.

Now let's consider other functions of the form $(f(x)/g(x))$ where we get the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ and determine an appropriate analytical method for computing the limits.

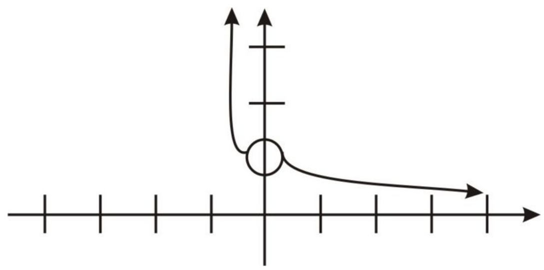
Example 1:

Consider the function $f(x) = \frac{\ln(x+1)}{x}$ and suppose we wish to find $\lim_{x \to 0} \frac{\ln(x+1)}{x}$ and $\lim_{x \to \infty} \frac{\ln(x+1)}{x}$ We note the following:

1. Direct substitution leads to the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
2. The function in the numerator is not a polynomial function, so we cannot use our previous methods such as applying $\lim_{x \to \infty} \frac{1}{x^p} = 0$.

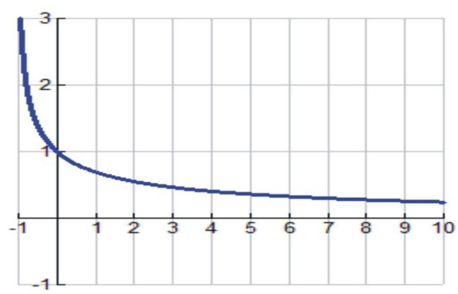
Let's examine both the graph and values of the function for appropriate x values, to see if they cluster around particular y values. Here is a sketch of the graph and a table of extreme values.

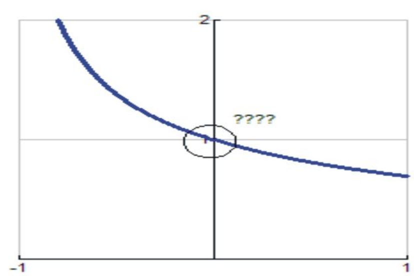
We first note that domain of the function is $(-1,0) \cup (0,+\infty)$ and is indicated in the graph as follows:



So, $\lim_{x \to 0} \frac{\ln(x+1)}{x}$ appears to approach the value 1 as the following table suggests.

Note: Please see Differentiation and Integration of Logarithmic and Exponential Functions in Chapter 6 for more on derivatives of Logarithmic functions.





x	$\ln(x+1)/x$
-0.1	1.05361
-0.001	1.0005
0	undef
0.001	0.9995
0.1	0.953102

So we infer that $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$.

For the infinite limit, $\lim_{x\to\infty} \frac{\ln(x+1)}{x} = 1$, the inference of the limit is not as obvious. The function appears to approach the value 0 but does so very slowly, as the following table suggests.

x	$\ln(x+1)/x$
10	0.23979
50	0.078637
100	0.046151
1000	0.006909
10000	0.000921

This unpredictable situation will apply to many other functions of the form. Hence we need another method that will provide a different tool for analyzing functions of the form $\frac{f(x)}{g(x)}$.

L'Hospital's Rule: Let functions f and g be differentiable at every number other than c in some interval, with $g'(x) \neq 0$ if $x \neq c$. If $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, or if $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$, then:

1. $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ as long as this latter limit exists or is infinite.
2. If f and g are differentiable at every number x greater than some number a, with $g'(x) \neq 0$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ as long as this latter limit exists or is infinite.

Let's look at applying the rule to some examples.

Example 2:

We will start by reconsidering the previous example, $f(x) = \frac{\ln(x+1)}{x}$, and verify the following limits using L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1.$$
$$\lim_{x\to\infty} \frac{\ln(x+1)}{x} = 0.$$

Solution:

Since $\lim_{x\to 0} \ln(x+1) = \lim_{x\to 0} x = 0$, L'Hospital's Rule applies and we have

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{1} = \frac{1}{1} = 1.$$

Likewise,

$$\lim_{x\to\infty} \frac{\ln(x+1)}{x} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{1} = \frac{0}{1} = 0.$$

Now let's look at some more examples.

Example 3:

Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Solution:

Since $\lim_{x \to 0}(e^x - 1) = \lim_{x \to 0} x = 0$, L'Hospital's Rule applies and we have

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = \frac{1}{1} = 1.$$

Example 4:

Evaluate $\lim_{x \to +\infty} \frac{x^2}{e^x}$

Solution:

Since $\lim_{x \to +\infty} x^2 = \lim_{x \to +\infty} e^x = +\infty$, L'Hospital's Rule applies and we have

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x}.$$

Here we observe that we still have the indeterminate form $\frac{\infty}{\infty}$. So we apply L'Hospital's Rule again to find the limit as follows:

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0$$

L'Hospital's Rule can be used repeatedly on functions like this. It is often useful because polynomial functions can be reduced to a constant.

Let's look at an example with trigonometric functions.

Example 5:

Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

Solution:

Since $\lim_{x \to 0}(1 - \cos x) = \lim_{x \to 0} x^2 = 0$, L'Hospital's Rule applies and we have

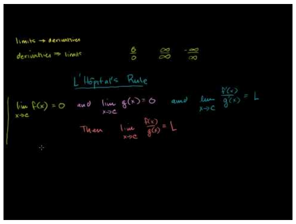
$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Lesson Summary

1. We learned to examine end behavior of functions on infinite intervals.
2. We determined horizontal asymptotes of rational functions.
3. We examined indeterminate forms of limits of rational functions.
4. We applied L'Hospital's Rule to find limits of rational functions.

Multimedia Links

For an introduction to L'Hopital's Rule **(8.0)**, see Khan Academy, L'Hopital's Rule (8:51).



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Review Questions

1. Use your graphing calculator to estimate $\lim_{x \to +\infty} [x[\ln(x+3) - \ln(x)]]$.
2. Use your graphing calculator to estimate $\lim_{x \to +\infty} \frac{x}{\ln(1+2e^x)}$.

In problems #3–10, use L'Hospital's Rule to compute the limits, if they exist.

3. $\lim_{x \to 3} \frac{x^2-9}{x-3}$
4. $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$
5. $\lim_{x \to +\infty} \frac{\ln(x)}{\sqrt{x}}$
6. $\lim_{x \to +\infty} x^2 e^{-2x}$
7. $\lim_{x \to 0} (1-x)^{\frac{1}{x}}$
8. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$
9. $\lim_{x \to -\infty} \frac{e^x - 1 - x}{x^2}$
10. $\lim_{x \to \infty} x^{-\frac{1}{4}} \ln(x)$

3.6 Analyzing the Graph of a Function

Learning Objectives

A student will be able to:

- Summarize the properties of function including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, limits at infinity.
- Apply the First and Second Derivative Tests to sketch graphs.

Introduction

In this lesson we summarize what we have learned about using derivatives to analyze the graphs of functions. We will demonstrate how these various methods can be applied to help us examine a function's behavior and sketch its graph. Since we have already discussed the various techniques, this lesson will provide examples of using the techniques to analyze the examples of representative functions we introduced in the Lesson on Relations and Functions, particularly rational, polynomial, radical, and trigonometric functions. Before we begin our work on these examples, it may be useful to summarize the kind of information about functions we now can generate based on our previous discussions. Let's summarize our results in a table like the one shown because it provides a useful template with which to organize our findings.

TABLE 3.3: Table Summary

$f(x)$	Analysis
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

Example 1: ***Analyzing Rational Functions***

Consider the function $f(x) = \frac{x^2-4}{x^2-2x-8}$.

General Properties: The function appears to have zeros at $x = \pm 2$. However, once we factor the expression we see

$$\frac{x^2-4}{x^2-2x-8} = \frac{(x+2)(x-2)}{(x-4)(x+2)} = \frac{x-2}{x-4}.$$

Hence, the function has a zero at $x = 2$, there is a hole in the graph at $x = -2$, the domain is $(-\infty, -2) \cup (-2, 4) \cup$

$(4, +\infty)$, and the $y-$intercept is at $(0, \frac{1}{2})$.

Asymptotes and Limits at Infinity

Given the domain, we note that there is a vertical asymptote at $x = 4$. To determine other asymptotes, we examine the limit of f as $x \to \infty$ and $x \to -\infty$. We have

$$\lim_{x\to\infty} \frac{x^2-4}{x^2-2x-8} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{1-\frac{2}{x}-\frac{8}{x^2}} = 1.$$

Similarly, we see that $\lim_{x\to-\infty} \frac{x^2-4}{x^2-2x-8} = 1$. We also note that $y \neq \frac{2}{3}$ since $x \neq -2$.

Hence we have a horizontal asymptote at $y = 1$.

Differentiability

$f'(x) = \frac{-2x^2-8x-8}{(x^2-2x-8)} = \frac{-2}{(x-4)^2} < 0$. Hence the function is differentiable at every point of its domain, and since $f'(x) < 0$ on its domain, then f is decreasing on its domain, $(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$.

$$f''(x) = \frac{4}{(x-4)^3}.$$

$f''(x) \neq 0$ in the domain of f. Hence there are no relative extrema and no inflection points.

So $f''(x) > 0$ when $x > 4$. Hence the graph is concave up for $x > 4$.

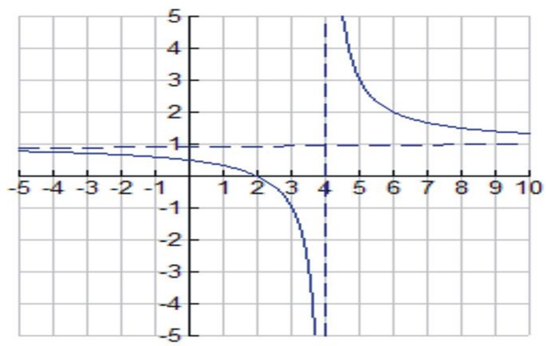
Similarly, $f''(x) < 0$ when $x < 4$. Hence the graph is concave down for $x < 4, x \neq -2$.

Let's summarize our results in the table before we sketch the graph.

TABLE 3.4: Table Summary

$f(x) = \frac{x^2-4}{x^2-2x-8}$	*Analysis*
Domain and Range	$D = (-\infty, -2) \cup (-2, 4) \cup (4, +\infty) R = \{\text{all reals} \neq 1 \text{ or } \frac{2}{3}\}$
Intercepts and Zeros	zero at $x = 2$, $y-$intercept at $\left(0, \frac{1}{2}\right)$
Asymptotes and limits at infinity	VA at $x = 4$, HA at $y = 1$, hole in the graph at $x = -2$
Differentiability	differentiable at every point of its domain
Intervals where f is increasing	nowhere
Intervals where f is decreasing	$(-\infty, -2) \cup (-2, 4) \cup (4, +\infty)$
Relative extrema	none
Concavity	concave up in $(4, +\infty)$, concave down in $(-\infty, -2) \cup (-2, 4)$
Inflection points	none

Finally, we sketch the graph as follows:



Let's look at examples of the other representative functions we introduced in Lesson 1.2.

Example 2:

Analyzing Polynomial Functions

Consider the function $f(x) = x^3 + 2x^2 - x - 2$.

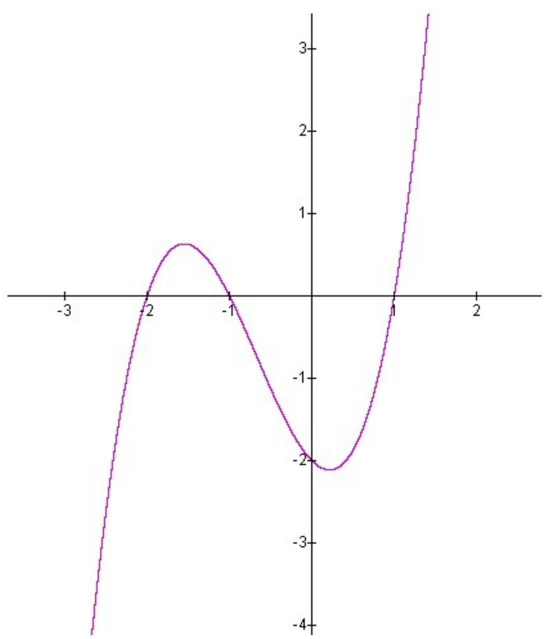
General Properties

The domain of f is $(-\infty, +\infty)$ and the $y-$intercept at $(0, -2)$.

The function can be factored

$$f(x) = x^3 + 2x^2 - x - 2 = x^2(x+2) - 1(x+2) = (x^2 - 1)(x+2) = (x-1)(x+1)(x+2)$$

and thus has zeros at $x = \pm 1, -2$.



Asymptotes and limits at infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Differentiability

$f'(x) = 3x^2 + 4x - 1 = 0$ if $x = \frac{-4\pm\sqrt{28}}{6} = \frac{-2\pm\sqrt{7}}{3}$. These are the critical values. We note that the function is differentiable at every point of its domain.

$f'(x) > 0$ on $\left(-\infty, \frac{-2-\sqrt{7}}{3}\right)$ and $\left(\frac{-2+\sqrt{7}}{3}, +\infty\right)$; hence the function is increasing in these intervals.

Similarly, $f'(x) < 0$ on $\left(\frac{-2-\sqrt{7}}{3}, \frac{-2+\sqrt{7}}{3}\right)$ and thus is f decreasing there.

$f''(x) = 6x + 4 = 0$ if $x = -\frac{2}{3}$, where there is an inflection point.

In addition, $f''\left(\frac{-2-\sqrt{7}}{3}\right) < 0$. Hence the graph has a relative maximum at $x = \frac{-2-\sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$.

We note that $f''(x) < 0$ for $x < -\frac{2}{3}$. The graph is concave down in $\left(-\infty, -\frac{2}{3}\right)$.

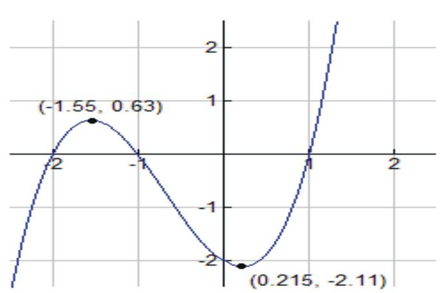
And we have $f''\left(\frac{-2+\sqrt{7}}{3}\right) > 0$; hence the graph has a relative minimum at $x = \frac{-2+\sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.

We note that $f''(x) > 0$ for $x > -\frac{2}{3}$. The graph is concave up in $\left(-\frac{2}{3}, +\infty\right)$.

TABLE 3.5: Table Summary

$f(x) = x^3 + 2x^2 - x - 2$	*Analysis*
Domain and Range	$D = (-\infty, +\infty), R = \{$ all reals $\}$
Intercepts and Zeros	zeros at $x = \pm 1, -2, y$, intercept at $(0, -2)$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable at every point of it's domain
Intervals where f is increasing	$\left(-\infty, \frac{-2-\sqrt{7}}{3}\right)$ and $\left(\frac{-2+\sqrt{7}}{3}, +\infty\right)$
Intervals where f is decreasing	$\left(\frac{-2-\sqrt{7}}{3}, \frac{-2+\sqrt{7}}{3}\right)$
Relative extrema	relative maximum at $x = \frac{-2-\sqrt{7}}{3}$ and located at the point $(-1.55, 0.63)$; relative minimum at $x = \frac{-2+\sqrt{7}}{3}$ and located at the point $(0.22, -2.11)$.
Concavity	concave up in $\left(-\frac{2}{3}, +\infty\right)$. concave down in $\left(-\infty, -\frac{2}{3}\right)$.
Inflection points	$x = -\frac{2}{3}$, located at the point $\left(-\frac{2}{3}, -.74\right)$

Here is a sketch of the graph:



Example 3: ***Analyzing Radical Functions***

Consider the function $f(x) = \sqrt{2x-1}$.

General Properties

The domain of f is $\left(\frac{1}{2}, +\infty\right)$, and it has a zero at $x = \frac{1}{2}$.

Asymptotes and Limits at Infinity

Given the domain, we note that there are no vertical asymptotes. We note that $\lim_{x\to\infty} f(x) = +\infty$.

Differentiability

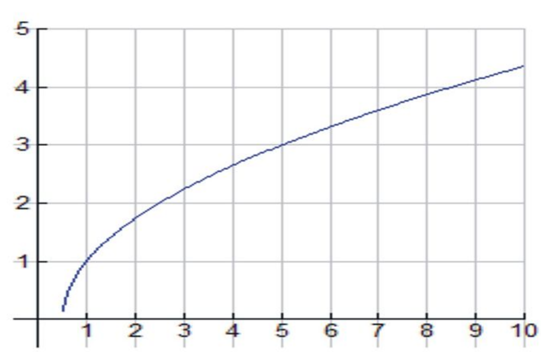
$f'(x) = \frac{1}{\sqrt{2x-1}} > 0$ for the entire domain of f. Hence f is increasing everywhere in its domain. $f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is a critical value.

$f''(x) = \frac{-1}{\sqrt{(2x-1)^3}} < 0$ everywhere in $\left(\frac{1}{2}, +\infty\right)$. Hence f is concave down in $\left(\frac{1}{2}, +\infty\right)$.$f'(x)$ is not defined at $x = \frac{1}{2}$, so $x = \frac{1}{2}$ is an absolute minimum.

TABLE 3.6: Table Summary

$f(x) = \sqrt{2x-1}$	*Analysis*
Domain and Range	$D = \left(\frac{1}{2}, +\infty\right), R = \{y \geq 0\}$
Intercepts and Zeros	zeros at $x = \frac{1}{2}$, no $y-$intercept
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $\left(\frac{1}{2}, +\infty\right)$
Intervals where f is increasing	everywhere in $D = \left(\frac{1}{2}, +\infty\right)$
Intervals where f is decreasing	nowhere
Relative extrema	none
	absolute minimum at $x = \frac{1}{2}$, located at $\left(\frac{1}{2}, 0\right)$
Concavity	concave down in $\left(\frac{1}{2}, +\infty\right)$
Inflection points	none

Here is a sketch of the graph:



Example 4: ***Analyzing Trigonometric Functions***

We will see that while trigonometric functions can be analyzed using what we know about derivatives, they will provide some interesting challenges that we will need to address. Consider the function $f(x) = x - 2\sin x$ on the interval $[-\pi, \pi]$.

General Properties

We note that f is a continuous function and thus attains an absolute maximum and minimum in $[-\pi,\pi]$. Its domain is $[-\pi,\pi]$ and its range is $R = \{-\pi \le y \le \pi\}$.

Differentiability

$f'(x) = 1 - 2\cos x = 0$ at $x = -\frac{\pi}{3}, \frac{\pi}{3}$.

Note that $f'(x) > 0$ on $\left(\frac{\pi}{3},\pi\right)$ and $\left(-\pi,-\frac{\pi}{3}\right)$; therefore the function is increasing in $\left(\frac{\pi}{3},\pi\right)$ and $\left(-\pi,-\frac{\pi}{3}\right)$.

Note that $f'(x) < 0$ on $\left(-\frac{\pi}{3},\frac{\pi}{3}\right)$; therefore the function is decreasing in $\left(-\frac{\pi}{3},\frac{\pi}{3}\right)$.

$f''(x) = 2\sin x = 0$ if $x = 0,\pi,-\pi$. Hence the critical values are at $x = -\pi,-\frac{\pi}{3},\frac{\pi}{3},$ and π.

$f''\left(\frac{\pi}{3}\right) > 0$; hence there is a relative minimum at $x = \frac{\pi}{3}$.

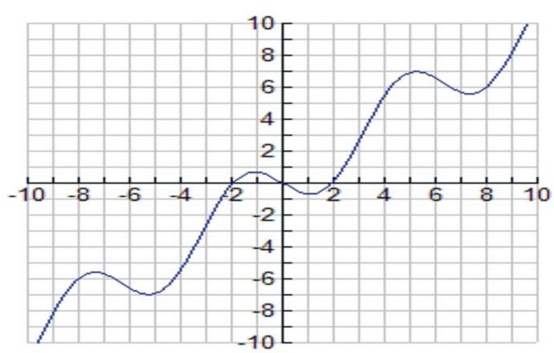
$f''\left(-\frac{\pi}{3}\right) < 0$; hence there is a relative maximum at $x = -\frac{\pi}{3}$.

$f''(x) > 0$ on $(0,\pi)$ and $f''(x) < 0$ on $(-\pi,0)$. Hence the graph is concave up and decreasing on $(0,\pi)$ and concave down on $(-\pi,0)$. There is an inflection point at $x = 0$, located at the point $(0,0)$.

Finally, there is absolute minimum at $x = -\pi$, located at $(-\pi,-\pi)$, and an absolute maximum at $x = \pi$, located at (π,π).

TABLE 3.7: Table Summary

$f(x) = x - 2\sin x$	*Analysis*
Domain and Range	$D = [-\pi,\pi], R = \{-\pi \le y \le \pi\}$
Intercepts and Zeros	$x = -\frac{\pi}{3}, \frac{\pi}{3}$
Asymptotes and limits at infinity	no asymptotes
Differentiability	differentiable in $D = [-\pi,\pi]$
Intervals where f is increasing	$\left(\frac{\pi}{3},\pi\right)$ and $\left(-\pi,-\frac{\pi}{3}\right)$
Intervals where f is decreasing	$\left(-\frac{\pi}{3},\frac{\pi}{3}\right)$
Relative extrema	relative maximum at $x = -\pi/3$
	relative minimum at $x = \pi/3$
	absolute maximum at $x = \pi$, located at (π,π)
	absolute minimum at $x = -\pi$, located at $(-\pi,-\pi)$
Concavity	concave up in $(0,\pi)$
Inflection points	$x = 0$, located at the point $(0,0)$

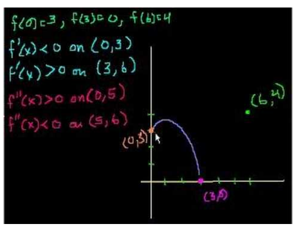


Lesson Summary

1. We summarized the properties of functions, including intercepts, domain, range, continuity, asymptotes, relative extreme, concavity, points of inflection, and limits at infinity.
2. We applied the First and Second Derivative Tests to sketch graphs.

Multimedia Links

Each of the problems above started with a function and then we analyzed its zeros, derivative, and concavity. Even without the function definition it is possible to sketch the graph if you know some key pieces of information. In the following video the narrator illustrates how to use information about the derivative of a function and given points on the function graph to sketch the function. Khan Academy Graphing with Calculus (9:44).



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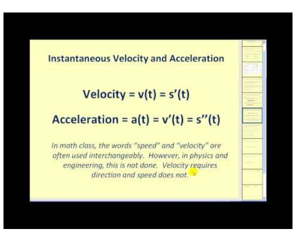
Another approach to this analysis is to look at a function, its derivative, and its second derivative on the same set of axes. This interactive applet called Curve Analysis allows you to trace function points on a graph and its first and second derivative. You can also enter new functions (including the ones from the examples above) to analyze the functions and their derivatives.

For more information about computing derivatives of higher orders **(7.0)**, see Math Video Tutorials by James Sousa, Higher-Order Derivatives: Part 1 of2 (7:34)



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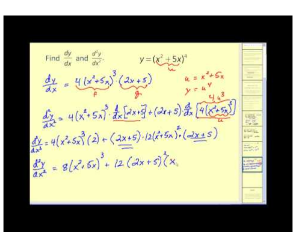
and Math Video Tutorials by James Sousa, Higher-Order Derivatives: Part 2 of2 (5:21).



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Review Questions

Summarize each of the following functions by filling out the table. Use the information to sketch a graph of the function.

1. $f(x) = x^3 + 3x^2 - x - 3$

TABLE 3.8:

$f(x) = x^3 + 3x^2 - x - 3$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

2. $f(x) = -x^4 + 4x^3 - 4x^2$

TABLE 3.9:

$f(x) = -x^4 + 4x^3 - 4x^2$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

3. $f(x) = \frac{2x-2}{x^2}$

TABLE 3.10:

$f(x) = \frac{2x-2}{x^2}$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	

TABLE 3.10: (continued)

$f(x) = \frac{2x-2}{x^2}$	*Analysis*
Inflection points	

4. $f(x) = x - x^{\frac{1}{3}}$

TABLE 3.11:

$f(x) = x - x^{\frac{1}{3}}$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

5. $f(x) = -\sqrt{2x-6} + 3$

TABLE 3.12:

$f(x) = -\sqrt{2x-6} + 3$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

6. $f(x) = x^2 - 2\sqrt{x}$

TABLE 3.13:

$f(x) = x^2 - 2\sqrt{x}$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

7. $f(x) = 1 + \cos x$ *on the interval* $[-\pi, \pi]$

TABLE 3.14:

$f(x) = 1 + \cos x$	*Analysis*
Domain and Range	
Intercepts and Zeros	
Asymptotes and limits at infinity	
Differentiability	
Intervals where f is increasing	
Intervals where f is decreasing	
Relative extrema	
Concavity	
Inflection points	

3.7 Optimization

Learning Objectives

A student will be able to:

- Use the First and Second Derivative Tests to find absolute maximum and minimum values of a function.
- Use the First and Second Derivative Tests to solve optimization applications.

Introduction

In this lesson we wish to extend our discussion of extrema and look at the absolute maximum and minimum values of functions. We will then solve some applications using these methods to maximize and minimize functions.

Absolute Maximum and Minimum

We begin with an observation about finding absolute maximum and minimum values of functions that are continuous on a closed interval. Suppose that f is continuous on a closed interval $[a,b]$. Recall that we can find relative minima and maxima by identifying the critical numbers of f in (a,b) and then applying the Second Derivative Test. The absolute maximum and minimum must come from either the relative extrema of f in (a,b) or the value of the function at the endpoints, $f(a)$ or $f(b)$. Hence the absolute maximum or minimum values of a function f that is continuous on a closed interval $[a,b]$ can be found as follows:

1. Find the values of f for each critical value in (a,b);
2. Find the values of the function f at the endpoints of $[a,b]$;
3. The absolute maximum will be the largest value of the numbers found in 1 and 2; the absolute minimum will be the smallest number.

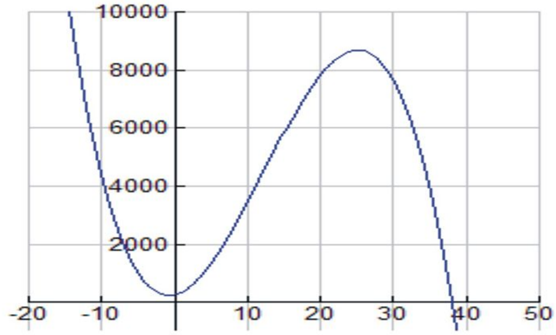
The optimization problems we will solve will involve a process of maximizing and minimizing functions. Since most problems will involve real applications that one finds in everyday life, we need to discuss how the properties of everyday applications will affect the more theoretical methods we have developed in our analysis. Let's start with the following example.

Example 1:

A company makes high-quality bicycle tires for both recreational and racing riders. The number of tires that the company sells is a function of the price charged and can be modeled by the formula $T(x) = -x^3 + 36.5x^2 + 50x + 250$, where x is the priced charged for each tire in dollars. At what price is the maximum number of tires sold? How many tires will be sold at that maximum price?

Solution:

Let's first look at a graph and make some observations. Set the viewing window ranges on your graphing calculator to $[-10, 50]$ for x and $[-500, 10000]$ for y. The graph should appear as follows:



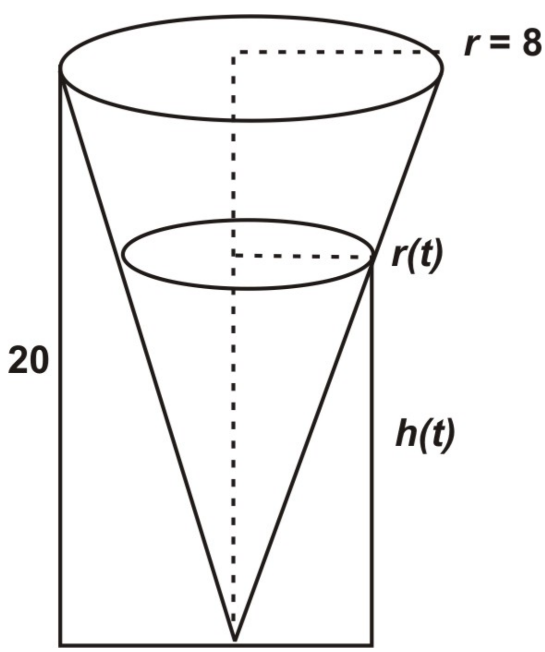
We first note that since this is a real-life application, we observe that both quantities, x and $T(x)$, are positive or else the problem makes no sense. These conditions, together with the fact that the zero of $T(x)$ is located at $x = 37.9$, suggest that the actual domain of this function is $0 < \times < 37$. This domain, which we refer to as a ***feasible domain,*** illustrates a common feature of optimization problems: that the real-life conditions of the situation under study dictate the domain values. Once we make this observation, we can use our First and Second Derivative Tests and the method for finding absolute maximums and minimums on a closed interval (in this problem, $[0,37]$), to see that the function attains an absolute maximum at $x = 25$, at the point $(25, 8687.5)$. So, charging a price of \$25 will result in a total of 8687 tires being sold.

In addition to the feasible domain issue illustrated in the previous example, many optimization problems involve other issues such as information from multiple sources that we will need to address in order to solve these problems. The next section illustrates this fact.

Primary and Secondary Equations

We will often have information from at least two sources that will require us to make some transformations in order to answer the questions we are faced with. To illustrate this, let's return to our Lesson on Related Rates problems and recall the right circular cone volume problem.

$$V = \frac{1}{3}\pi r^2 h.$$



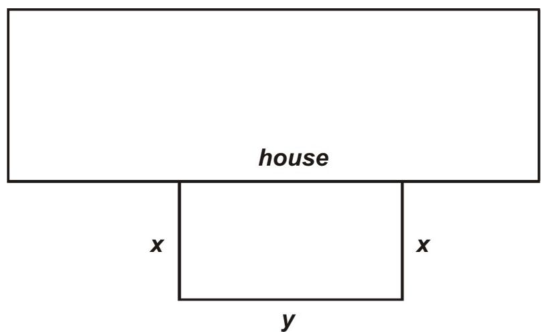
We started with the general volume formula $V = \frac{1}{3}\pi r^2 h$, but quickly realized that we did not have sufficient information to find $\frac{dh}{dt}$ since we had no information about the radius when the water level was at a particular height. So we needed to employ some indirect reasoning to find a relationship between r and h, $r(t) = \frac{2h(t)}{5}$. We then made an appropriate substitution in the original formula $(V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3)$ and were able to find the solution.

We started with a ***primary equation***, $V = \frac{1}{3}\pi r^2 h$, that involved two variables and provided a general model of the situation. However, in order to solve the problem, we needed to generate a ***secondary equation,*** $r(t) = \frac{2h(t)}{5}$, that we then substituted into the primary equation. We will face this same situation in most optimization problems.

Let's illustrate the situation with an example.

Example 2:

Suppose that Mary wishes to make an outdoor rectangular pen for her pet chihuahua. She would like the pen to enclose an area in her backyard with one of the sides of the rectangle made by the side of Mary's house as indicated in the following figure. If she has 90 ft of fencing to work with, what dimensions of the pen will result in the maximum area?



Solution:

The primary equation is the function that models the area of the pen and that we wish to maximize,

$$A = xy.$$

The secondary equation comes from the information concerning the fencing Mary has to work with. In particular,

$$2x + y = 90.$$

Solving for y we have

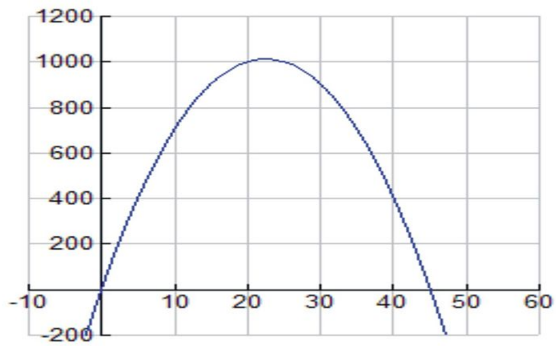
$$y = 90 - 2x.$$

We now substitute into the primary equation to get

$A = xy = x(90 - 2x)$, or

$$A = 90x - 2x^2.$$

It is always helpful to view the graph of the function to be optimized. Set the viewing window ranges on your graphing calculator to $[-10, 100]$ for x and $[-500, 1200]$ for y. The graph should appear as follows:



The feasible domain of this function is $0 < \times < 45$, which makes sense because if x is 45 feet, then the figure will be two 45-foot-long fences going away from the house with 0 feet left for the width, y. Using our First and Second Derivative Tests and the method for finding absolute maximums and minimums on a closed interval (in this problem, $[0,45]$), we see that the function attains an absolute maximum at $x = 22.5$, at the point $(22.5, 1012.5)$. So the dimensions of the pen should be $x = 22.5, y = 45$; with those dimensions, the pen will enclose an area of 1012.5 ft^2.

Recall in the Lesson Related Rates that we solved problems that involved a variety of geometric shapes. Let's consider a problem about surface areas of cylinders.

Example 3:

A certain brand of lemonade sells its product in $16-$ounce aluminum cans that hold $473 \text{ ml}(1 \text{ ml} = 1 \text{ cm}^3)$. Find the dimensions of the cylindrical can that will use the least amount of aluminum.

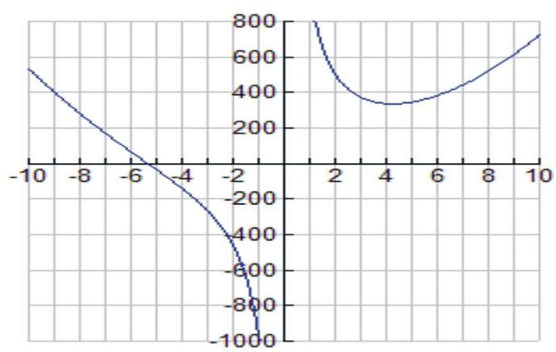
Solution:

We need to develop the formula for the surface area of the can. This consists of the top and bottom areas, each πr^2, and the surface area of the side, $2\pi rh$ (treating the side as a rectangle, the lateral area is (circumference of the top) $\times$ (height)). Hence the primary equation is

$$A = 2\pi r^2 + 2\pi rh.$$

We observe that both our feasible domains require $r, h > 0$.

In order to generate the secondary equation, we note that the volume for a circular cylinder is given by $V = \pi r^2 h$. Using the given information we can find a relationship between r and $h, h = \frac{473}{\pi r^2}$. We substitute this value into the primary equation to get $A = 2\pi r^2 + 2\pi r\left(\frac{473}{\pi r^2}\right)$, or $A = 2\pi r^2 + \frac{946}{r}$.



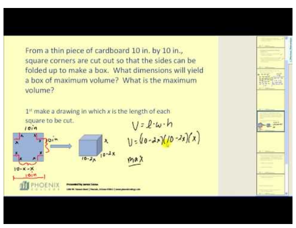
$\frac{dA}{dr} = 4\pi r - \frac{946}{r^2} = 0$ when $r = \sqrt[3]{\frac{946}{4\pi}} \approx 9.06$ cm. We note that $\frac{d^2A}{dr^2} > 0$ since $r > 0$. Hence we have a minimum surface area when $r = \sqrt[3]{\frac{946}{4\pi 0}} \approx 4.22$ cm and $h = \frac{473}{\pi(\sqrt[3]{\frac{946}{4\pi}})^2} = 8.44$ cm.

Lesson Summary

1. We used the First and Second Derivative Tests to find absolute maximum and minimum values of a function.
2. We used the First and Second Derivative Tests to solve optimization applications.

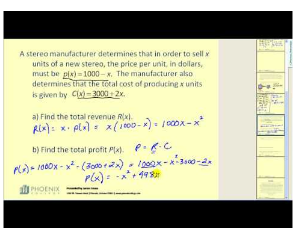
Multimedia Links

For video presentations of maximum-minimum Business and Economics applications **(11.0)**, see Math Video Tutorials by James Sousa, Max & Min Apps. w/calculus, Part 1 (9:57)



MEDIA
Click image to the left for more content.

and Math Video Tutorials by James Sousa, Max & Min Apps. w/calculus, Part 2 (4:51).



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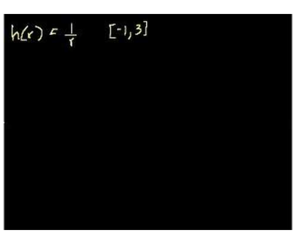
Click image to the left for more content.

To see more examples of worked out problems involving finding minima and maxima on an interval **(11.0)**, see the video at Khan Academy Minimum and Maximum Values on an Interval (11:42).



MEDIA
Click image to the left for more content.

This video shows the process of applying the first derivative test to problems with no context, just a given function and a domain. A classic problem in calculus involves maximizing the volume of an open box made by cutting squares from a rectangular sheet and folding up the edges. This very cool calculus applet shows one solution to this problem and multiple representations of the problem as well. Calculus Applet on Optimization

Review Questions

In problems #1–4, find the absolute maximum and absolute minimum values, if they exist.

1. $f(x) = 2x^2 - 6x + 6$ on $[0,5]$
2. $f(x) = x^3 + 3x^2$ on $[-2,3]$
3. $f(x) = 3x^{\frac{2}{3}} - 6x + 6$ on $[1,8]$
4. $f(x) = x^4 - x^3$ on $[-2,2]$
5. Find the dimensions of a rectangle having area 2000 ft^2 whose perimeter is as small as possible.
6. Find two numbers whose product is 50 and whose sum is a minimum.
7. John is shooting a basketball from half-court. It is approximately 45 ft from the half court line to the hoop. The function $s(t) = -0.025t^2 + t + 15$ models the basketball's height above the ground $s(t)$ in feet, when it is t feet from the hoop. How many feet from John will the ball reach its highest height? What is that height?
8. The height of a model rocket t seconds into flight is given by the formula $h(t) = -\frac{1}{3}t^3 + 4t^2 + 25t + 4$.
   a. How long will it take for the rocket to attain its maximum height?
   b. What is the maximum height that the rocket will reach?
   c. How long will the flight last?
9. Show that of all rectangles of a given perimeter, the rectangle with the greatest area is a square.
10. Show that of all rectangles of a given area, the rectangle with the smallest perimeter is a square.

3.8 Approximation Errors

Learning Objectives

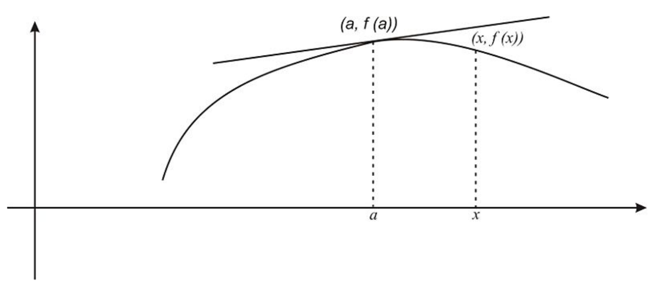
A student will be able to:

- Extend the Mean Value Theorem to make linear approximations.
- Analyze errors in linear approximations.
- Extend the Mean Value Theorem to make quadratic approximations.
- Analyze errors in quadratic approximations.

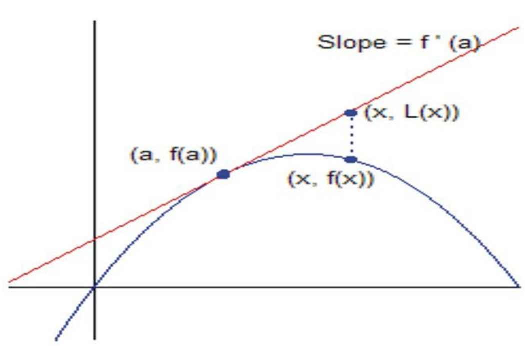
Introduction

In this lesson we will use the Mean Value Theorem to make approximations of functions. We will apply the Theorem directly to make linear approximations and then extend the Theorem to make quadratic approximations of functions.

Let's consider the tangent line to the graph of a function f at the point $(a, f(a))$. The equation of this line is $y = f(a) + f'(a)(x-a)$. We observe from the graph that as we consider x near a, the value of $f(x)$ is very close to $f(a)$.



In other words, for x values close to a, the tangent line to the graph of a function f at the point $(a, f(a))$ provides an approximation of $f(x)$ or $f(x) \approx f(a) + f'(a)(x-a)$. We call this the ***linear*** or ***tangent line approximation*** of f at a and indicate it by the formula $L(x) = f(a) + f'(a)(x-a)$.



The linear approximation can be used to approximate functional values that deviate slightly from known values. The following example illustrates this process.

Example 1:

Use the linear approximation of the function $f(x) = \sqrt{x-2}$ at $a = 6$ to approximate $\sqrt{3.95}$.

Solution:

We know that $f(6) = \sqrt{6-2} = 2$. So we will find the linear approximation of the function and substitute x values close to 6.

$$L(x) = f(6) + f'(6)(x-6).$$

We note that $f'(x) = \frac{1}{2\sqrt{x-2}}, f'(6) = \frac{1}{4}$.

We also know that $f(6) = 2$.

By substitution, we have

$f(x) \approx f(6) + f'(6)(x-6)$ for x near 6.

Hence $L(x) = 2 + \frac{1}{4}(x-6) = \frac{1}{2} + \frac{1}{4}x$.

We observe that to approximate $\sqrt{3.95}$ we need to evaluate the linear approximation at 5.95, and we have $L(5.95) = \frac{1}{2} + \frac{1}{4}(5.95) = 1.9875$. If we were to compare this approximation to the actual value, $\sqrt{3.95} \approx 1.9874$, we see that it is a very good approximation.

If we observe a table of x values close to 6, we see how the approximations compare to the actual value.

$f(x) = \sqrt{x-2}$	x	$L(x) = \dfrac{1}{2} + \dfrac{1}{4}x$	Actual
$\sqrt{3.95}$	5.95	1.9875	1.9874
$\sqrt{3.99}$	5.99	1.9975	1.9974
$\sqrt{4}$	6	2	2
$\sqrt{4.1}$	6.01	2.0025	2.0024
$\sqrt{4.05}$	6.05	2.0125	2.0124

Setting Error Estimates

We would like to have confidence in the approximations we make. We therefore can choose the x values close to a to ensure that the errors are within acceptable boundaries. For the previous example, we saw that the values of $L(x)$ close to $a = 6$, gave very good approximations, all within 0.0001 of the actual value.

Example 2:

Let's suppose that for the previous example, we did not require such precision. Rather, suppose we wanted to find the range of x values close to 6 that we could choose to ensure that our approximations lie within 0.01 of the actual value.

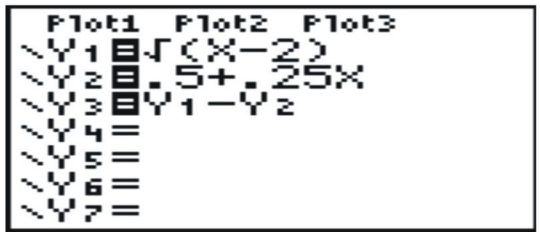
Solution:

The easiest way for us to find the proper range of x values is to use the graphing calculator. We first note that our precision requirement can be stated as $|\sqrt{x-2} - \left(\frac{1}{2} + \frac{x}{4}\right)| < 0.01$.

If we enter the functions $f(x) = \sqrt{x-2}$ and $L(x) = \frac{1}{2} + \frac{1}{4}x$ into the $Y =$ menu as Y_1 and Y_2, respectively, we will be able to view the function values of the functions using the **[TABLE]** feature of the calculator. In order to view the differences between the actual and approximate values, we can enter into the $Y =$ menu the difference function $Y_3 = Y_1 - Y_2$ as follows:

1. Go to the $Y =$ menu and place cursor on the Y_3 line.
2. Press the following sequence of key strokes: **[VARS] [FUNCTION]** $[Y_1]$. This will copy the function Y_1 onto the Y_3 line of the $Y =$ menu.
3. Press [-] to enter the subtraction operation onto the Y_3 line of the $Y =$ menu.
4. Repeat steps 1 - 2 and choose Y_2 to copy Y_2 onto the Y_3 line of the $Y =$ menu.

Your screen should now appear as follows:



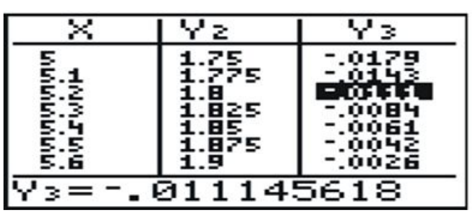
Now let's setup the **[TABLE]** function so that we find the required accuracy.

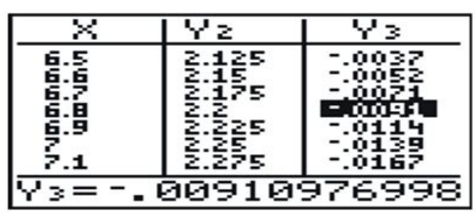
1. Press **2ND** followed by **[TBLSET]** to access the **Table Setup** screen.
2. Set the **[TBLStart]** value to 5 and $\triangle$ Tbl to 0.1.

Your screen should now appear as follows:

Now we are ready to find the required accuracy.

Access the **[TABLE]** function, scroll through the table, and find those x values that ensure $|Y_3| \leq 0.01$.





$5.3 \leq x \leq 6.8$

Non-Linear Approximations

It turns out that the linear approximations we have discussed are not the only approximations that we can derive using derivatives. We can use non-linear functions to make approximations. These are called ***Taylor Polynomials*** and are defined as

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n.$$

We call this the ***Taylor Polynomial of*** f ***centered at*** a.

For our discussion, we will focus on the quadratic case. The Taylor Polynomial corresponding to $n = 2$ is given by

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Note that this is just our linear approximation with an added term. Hence we can view it as an approximation of f for x values close to a.

Example 3:

Find the quadratic approximation of the function $f(x) = \sqrt{x-2}$ at $a = 6$ and compare them to the linear approximations from the first example.

Solution:

Recall that $L(x) = \frac{1}{2} + \frac{1}{4}x$.

Hence $T_2(x) = L(x) + \frac{1}{2}f''(6)(x-6)^2$.

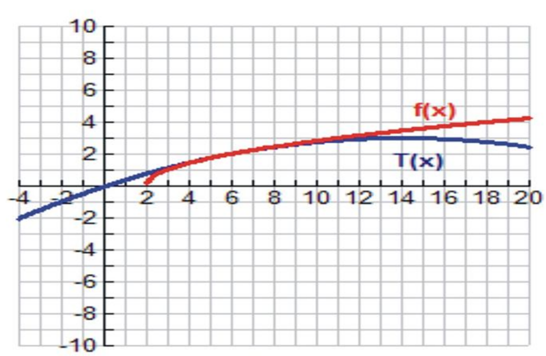
$f''(x) = -\frac{1}{4\sqrt{(x-2)^3}}$; so $f''(6) = -\frac{1}{4\sqrt{(6-2)^3}} = -\frac{1}{32}$.

Hence $T_2(x) = L(x) - \frac{1}{64}(x-6)^2 = \frac{1}{2} + \frac{1}{4}x - \frac{1}{64}(x-6)^2 = -\frac{1}{64}x^2 + \frac{7}{16}x - \frac{1}{16}$.

So $T_2(x) = -\frac{1}{64}x^2 + \frac{7}{16}x - \frac{1}{16}$. If we update our table from the first example we can see how the quadratic approximation compares with the linear approximation.

$f(x) = \sqrt{x-2}$	x	$L(x) = \frac{1}{2} + \frac{1}{4}x$	$T_2(x) = -\frac{1}{64}x^2 + \frac{7}{16}x - \frac{1}{16}$	Actual
$\sqrt{3.95}$	5.95	1.9875	1.9874	1.9874
$\sqrt{3.99}$	5.99	1.9975	1.9974	1.9974
$\sqrt{4}$	6	2	2	2
$\sqrt{4.1}$	6.01	2.0025	2.0024	2.0024
$\sqrt{4.05}$	6.05	2.0125	2.0124	2.0124

As you can see from the graph below, $T(x)$ is an excellent approximation of $f(x)$ near $x = 6$.



We get a slightly better approximation for the quadratic than for the linear. If we reflect on this a bit, the finding makes sense since the shape and properties of quadratic functions more closely approximate the shape of radical functions.

Finally, as in the first example, we wish to determine the range of x values that will ensure that our approximations are within 0.01 of the actual value. Using the **[TABLE]** feature of the calculator, we find that if $4.444 \leq \times \leq 7.87$, then $|\sqrt{x-2} - T(x)| < 0.01$.

Lesson Summary

1. We extended the Mean Value Theorem to make linear approximations.
2. We analyzed errors in linear approximations.
3. We extended the Mean Value Theorem to make quadratic approximations.
4. We analyzed errors in quadratic approximations.

Review Questions

In problems #1–4, find the linearization $L(x)$ of the function at $x = a$.

1. $f(x) = 2x^4 - 6x^3$ near $a = -2$
2. $f(x) = x^{\frac{2}{3}}$ near $a = 27$
3. Find the linearization of the function $f(x) = \sqrt{5-x}$ near a = 1 and use it to approximate $\sqrt{4.01}$.
4. Based on using linear approximations, is the following approximation reasonable?

$$1.001^4 = 1.004$$

5. Use a linear approximation to approximate the following:

$$16.08^{\frac{3}{4}}$$

6. Verify the the following linear approximation at $a = 1$. Determine the values of x for which the linear approximation is accurate to 0.01.

$$\sqrt[3]{2-x} \approx \frac{4}{3} - \frac{x}{3}$$

7. Find the quadratic approximation for the function in #3, $f(x) = \sqrt{5-x}$ near $a = 1$.
8. Determine the values of x for which the quadratic approximation found in #7 is accurate to 0.01.
9. Determine the quadratic approximation for $f(x) = 2x^4 - 6x^3$ near $a = -2$. Do you expect that the quadratic approximation is better or worse than the linear approximation? Explain your answer.

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9728 .

CHAPTER 4 Integration

Chapter Outline

4.1 Indefinite Integrals Calculus

Learning Objectives

A student will be able to:

- Find antiderivatives of functions.
- Represent antiderivatives.
- Interpret the constant of integration graphically.
- Solve differential equations.
- Use basic antidifferentiation techniques.
- Use basic integration rules.

Introduction

In this lesson we will introduce the idea of the ***antiderivative*** of a function and formalize as ***indefinite integrals.*** We will derive a set of rules that will aid our computations as we solve problems.

Antiderivatives

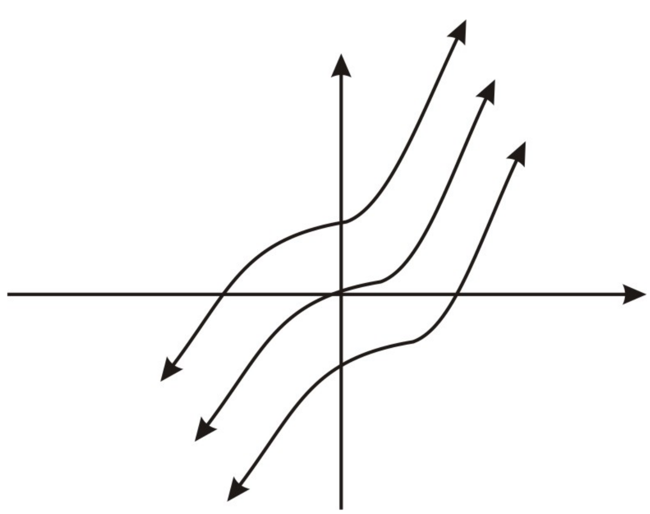
Definition

A function $F(x)$ is called an ***antiderivative*** of a function f if $F'(x) = f(x)$ for all x in the domain of f.

Example 1:

Consider the function $f(x) = 3x^2$. Can you think of a function $F(x)$ such that $F'(x) = f(x)$? ***(Answer:** $F(x) = x^3$,$F(x) = x^3 - 6$, **many other examples.)***

Since we differentiate $F(x)$ to get $f(x)$, we see that $F(x) = x^3 + C$ will work for any constant C. Graphically, we can think the set of all antiderivatives as vertical transformations of the graph of $F(x) = x^3$. The figure shows two such transformations.



With our definition and initial example, we now look to formalize the definition and develop some useful rules for computational purposes, and begin to see some applications.

Notation and Introduction to Indefinite Integrals

The process of finding antiderivatives is called ***antidifferentiation***, more commonly referred to as ***integration.*** We have a particular sign and set of symbols we use to indicate integration:

$$\int f(x)dx = F(x) + C.$$

We refer to the left side of the equation as "the indefinite integral of $f(x)$ with respect to x." The function $f(x)$ is called the ***integrand*** and the constant C is called the ***constant of integration.*** Finally the symbol dx indicates that we are to integrate with respect to x.

Using this notation, we would summarize the last example as follows:

$$\int 3x^2 dx = x^3 + C$$

Using Derivatives to Derive Basic Rules of Integration

As with differentiation, there are several useful rules that we can derive to aid our computations as we solve problems. The first of these is a rule for integrating power functions, $f(x) = x^n$ $[n \neq -1]$, and is stated as follows:

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C.$$

We can easily prove this rule. Let $F(x) = \frac{1}{n+1} x^{n+1} + C, n \neq -1$. We differentiate with respect to x and we have:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) + \frac{d}{dx}(C) \\ &= \left(\frac{1}{n+1}\right)\frac{d}{dx}\left(x^{n+1}\right) + \frac{d}{dx}(C) \\ &= \left(\frac{n+1}{n+1}\right)x^n + 0 \\ &= x^n. \end{aligned}$$

The rule holds for $f(x) = x^n [n \neq -1]$. What happens in the case where we have a power function to integrate with $n = -1$, say $\int x^{-1} dx = \int \frac{1}{x} dx$. We can see that the rule does not work since it would result in division by 0. However, if we pose the problem as finding $F(x)$ such that $F'(x) = \frac{1}{x}$, we recall that the derivative of logarithm functions had this form. In particular, $\frac{d}{dx} \ln x = \frac{1}{x}$. Hence

$$\int \frac{1}{x} dx = \ln x + C.$$

In addition to logarithm functions, we recall that the basic exponentional function, $f(x) = e^x$, was special in that its derivative was equal to itself. Hence we have

$$\int e^x dx = e^x + C.$$

Again we could easily prove this result by differentiating the right side of the equation above. The actual proof is left as an exercise to the student.

As with differentiation, we can develop several rules for dealing with a finite number of integrable functions. They are stated as follows:

If f and g are integrable functions, and C is a constant, then

$$\int [f(x) + g(x)]\,dx = \int f(x)dx + \int g(x)dx,$$
$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx,$$
$$\int [Cf(x)]dx = C\int f(x)dx.$$

Example 2:

Compute the following indefinite integral.

$$\int \left[2x^3 + \frac{3}{x^2} - \frac{1}{x}\right] dx.$$

Solution:

Using our rules we have

$$\begin{aligned}\int \left[2x^3 + \frac{3}{x^2} - -\frac{1}{x}\right] dx &= 2\int x^3 dx + 3\int \frac{1}{x^2}dx - \int \frac{1}{x}dx \\ &= 2\left(\frac{x^4}{4}\right) + 3\left(\frac{x^{-1}}{-1}\right) - \ln x + C \\ &= \frac{x^4}{2} - \frac{3}{x} - \ln x + C.\end{aligned}$$

Sometimes our rules need to be modified slightly due to operations with constants as is the case in the following example.

Example 3:

Compute the following indefinite integral:

$$\int e^{3x} dx.$$

Solution:

We first note that our rule for integrating exponential functions does not work here since $\frac{d}{dx}e^{3x} = 3e^{3x}$. However, if we remember to divide the original function by the constant then we get the correct antiderivative and have

$$\int e^{3x}dx = \frac{e^{3x}}{3} + C.$$

We can now re-state the rule in a more general form as

$$\int e^{kx}dx = \frac{e^{kx}}{k} + C.$$

Differential Equations

We conclude this lesson with some observations about integration of functions. First, recall that the integration process allows us to start with function f from which we find another function $F(x)$ such that $F'(x) = f(x)$. This latter equation is called a ***differential equation.*** This characterization of the basic situation for which integration applies gives rise to a set of equations that will be the focus of the Lesson on The Initial Value Problem.

Example 4:

Solve the general differential equation $f'(x) = x^{\frac{2}{3}} + \sqrt{x}$.

Solution:

We solve the equation by integrating the right side of the equation and have

$$f(x) = \int f'(x)dx = \int x^{\frac{2}{3}}dx + \int \sqrt{x}dx.$$

We can integrate both terms using the power rule, first noting that $\sqrt{x} = x^{\frac{1}{2}}$, and have

$$f(x) = \int x^{\frac{2}{3}}dx + \int x^{\frac{1}{2}}dx = \frac{3}{5}x^{\frac{5}{3}} + \frac{2}{3}x^{\frac{3}{2}} + C.$$

Lesson Summary

1. We learned to find antiderivatives of functions.
2. We learned to represent antiderivatives.
3. We interpreted constant of integration graphically.
4. We solved general differential equations.
5. We used basic antidifferentiation techniques to find integration rules.
6. We used basic integration rules to solve problems.

Multimedia Link

The following applet shows a graph, $f(x)$ and its derivative, $f'(x)$. This is similar to other applets we've explored with a function and its derivative graphed side-by-side, but this time $f(x)$ is on the right, and $f'(x)$ is on the left. If you edit the definition of $f'(x)$, you will see the graph of $f(x)$ change as well. The c parameter adds a constant to $f(x)$. Notice that you can change the value of c without affecting $f'(x)$. Why is this? Antiderivative Applet .

Review Questions

In problems #1–3, find an antiderivative of the function

1. $f(x) = 1 - 3x^2 - 6x$
2. $f(x) = x - x^{\frac{2}{3}}$
3. $f(x) = \sqrt[5]{2x+1}$

In #4–7, find the indefinite integral

4. $\int (2 + \sqrt{5})dx$
5. $\int 2(x-3)^3 dx$
6. $\int (x^2 \cdot \sqrt[3]{x})dx$
7. $\int \left(x + \frac{1}{x^4\sqrt{x}}\right) dx$
8. Solve the differential equation

$$f'(x) = 4x^3 - 3x^2 + x - 3$$

9. Find the antiderivative $F(x)$ of the function $f(x) = 2e^{2x} + x - 2$ that satisfies $F(0) = 5$.
10. Evaluate the indefinite integral $\int |x| dx$ (Hint: Examine the graph of $f(x) = |x|$.)

4.2 The Initial Value Problem

Learning Objectives

- Find general solutions of differential equations
- Use initial conditions to find particular solutions of differential equations

Introduction

In the Lesson on Indefinite Integrals Calculus we discussed how finding antiderivatives can be thought of as finding solutions to differential equations: $F'(x) = f(x)$. We now look to extend this discussion by looking at how we can designate and find particular solutions to differential equations.

Let's recall that a general differential equation will have an infinite number of solutions. We will look at one such equation and see how we can impose conditions that will specify exactly one particular solution.

Example 1:

Suppose we wish to solve the following equation:

$$f'(x) = e^{3x} - 6\sqrt{x}.$$

Solution:

We can solve the equation by integration and we have

$$f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + C.$$

We note that there are an infinite number of solutions. In some applications, we would like to designate exactly one solution. In order to do so, we need to impose a condition on the function f. We can do this by specifying the value of f for a particular value of x. In this problem, suppose that we add the condition that $f(0) = 1$. This will specify exactly one value of C and thus one particular solution of the original equation:

Substituting $f(0) = 1$ into our general solution $f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + C$ gives $1 = \frac{1}{3}e^{3(0)} - 4(0)^{\frac{3}{2}} + C$ or $C = 1 - \frac{1}{3} = \frac{2}{3}$. Hence the solution $f(x) = \frac{1}{3}e^{3x} - 4x^{\frac{3}{2}} + \frac{2}{3}$ is the ***particular solution*** of the original equation $f'(x) = e^{3x} - 6\sqrt{x}$ satisfying the ***initial condition*** $f(0) = 1$.

We now can think of other problems that can be stated as differential equations with initial conditions. Consider the following example.

Example 2:

Suppose the graph of f includes the point $(2,6)$ and that the slope of the tangent line to f at any point x is given by the expression $3x + 4$. Find $f(-2)$.

Solution:

We can re-state the problem in terms of a differential equation that satisfies an initial condition.

$f'(x) = 3x+4$ with $f(2) = 6$.

By integrating the right side of the differential equation we have

$$f(x) = \frac{3}{2}x^2 + 4x + C \text{ as the general solution. Substituting the condition that } f(2) = 6 \text{ gives}$$

$$6 = \frac{3}{2}(2)^2 + 4(2) + C,$$

$$6 = 6 + 8 + C,$$

$$C = -8.$$

Hence $f(x) = \frac{3}{2}x^2 + 4x - 8$ is the ***particular solution*** of the original equation $f'(x) = 3x+4$ satisfying the ***initial condition*** $f(2) = 6$.

Finally, since we are interested in the value $f(-2)$, we put -2 into our expression for f and obtain:

$$f(-2) = -10$$

Lesson Summary

1. We found general solutions of differential equations.
2. We used initial conditions to find particular solutions of differential equations.

Multimedia Link

The following applet allows you to set the initial equation for $f'(x)$ and then the slope field for that equation is displayed. In magenta you'll see one possible solution for $f(x)$. If you move the magenta point to the initial value, then you will see the graph of the solution to the initial value problem. Follow the directions on the page with the applet to explore this idea, and then try redoing the examples from this section on the applet. Slope Fields Applet .

Review Questions

In problems #1–3, solve the differential equation for $f(x)$.

1. $f'(x) = 2e^{2x} - 2\sqrt{x}$
2. $f'(x) = \sin x - \frac{1}{e^x}$
3. $f''(x) = (2+x)\sqrt{x}$

In problems #4–7, solve the differential equation for $f(x)$ given the initial condition.

4. $f'(x) = 6x^5 - 4x^2 + \frac{7}{3}$ and $f(1) = 4$

5. $f'(x) = 3x^2 + e^{2x}$ and $f(0) = 3$.
6. $f'(x) = \sqrt[3]{x^2} - \frac{1}{x^2}$ and $f(1) = 3$
7. $f'(x) = (2\cos x - \sin x), -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and $f\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{1}{2}$
8. Suppose the graph of f includes the point $(-2, 4)$ and that the slope of the tangent line to f at x is $-2x + 4$. Find $f(5)$.

In problems #9–10, find the function f that satisfies the given conditions.

9. $f''(x) = \sin x - e^{-2x}$ with $f'(0) = \frac{5}{2}$ and $f(0) = 0$
10. $f''(x) = \frac{1}{\sqrt{x}}$ with $f'(4) = 7$ and $f(4) = 25$

4.3 The Area Problem

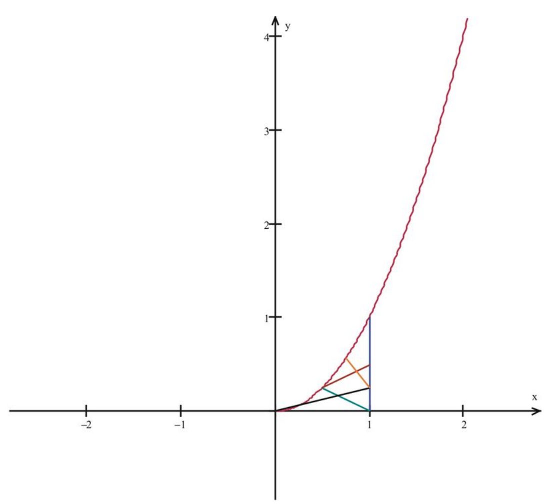
Learning Objectives

- Use sigma notation to evaluate sums of rectangular areas
- Find limits of upper and lower sums
- Use the limit definition of area to solve problems

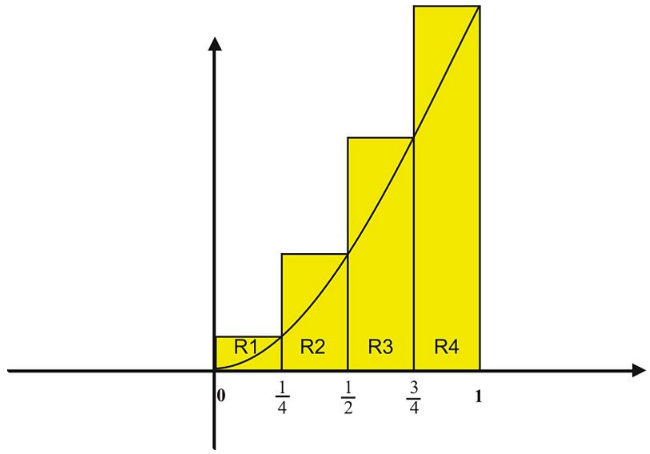
Introduction

In The Lesson The Calculus we introduced the area problem that we consider in integral calculus. The basic problem was this: $f(x) = x^2$.

Suppose we are interested in finding the area between the $x-$axis and the curve of $f(x) = x^2$, from $x = 0$ to $x = 1$.



We approximated the area by constructing four rectangles, with the height of each rectangle equal to the maximum value of the function in the sub-interval.



We then summed the areas of the rectangles as follows:

$$R_1 = \frac{1}{4} \cdot f\left(\frac{1}{4}\right) = \frac{1}{64},$$
$$R_2 = \frac{1}{4} \cdot f\left(\frac{1}{2}\right) = \frac{1}{16},$$
$$R_3 = \frac{1}{4} \cdot f\left(\frac{3}{4}\right) = \frac{9}{64},$$
$$R_4 = \frac{1}{4} \cdot f(1) = \frac{1}{4},$$

and $R_1 + R_2 + R_3 + R_4 = \frac{30}{64} = \frac{15}{32} \approx 0.46$.

We call this the ***upper sum*** since it is based on taking the maximum value of the function within each sub-interval. We noted that as we used more rectangles, our area approximation became more accurate.

We would like to formalize this approach for both upper and lower sums. First we note that the ***lower sums*** of the area of the rectangles results in $R_1 + R_2 + R_3 + R_4 = 13/64 \approx 0.20$ Our intuition tells us that the true area lies somewhere between these two sums, or $0.20 <$ Area < 0.46 and that we will get closer to it by using more and more rectangles in our approximation scheme.

In order to formalize the use of sums to compute areas, we will need some additional notation and terminology.

Sigma Notation

In The Lesson The Calculus we used a notation to indicate the upper sum when we increased our rectangles to $N = 16$ and found that our approximation $A = \frac{1432}{4096} \approx .34$. The notation we used to enabled us to indicate the sum without the need to write out all of the individual terms. We will make use of this notation as we develop more formal definitions of the area under the curve.

Let's be more precise with the notation. For example, the quantity $A = \sum R_i$ was found by summing the areas of $N = 16$ rectangles. We want to indicate this process, and we can do so by providing indices to the symbols used as follows:

$$A = \sum_{i=1}^{16} R_i = R_1 + R_2 + R_3 + ... + R_{15} + R_{16}.$$

The sigma symbol with these indices tells us how the rectangles are labeled and how many terms are in the sum.

Useful Summation Formulas

We can use the notation to indicate useful formulas that we will have occasion to use. For example, you may recall that the sum of the first n integers is $n(n+1)/2$. We can indicate this formula using sigma notation. The formula is given here along with two other formulas that will become useful to us.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

We can show from associative, commutative, and distributive laws for real numbers that

$\sum_{i=1}^{n}(a_i+b_i)=\sum_{i=1}^{n}(a_i)+\sum_{i=1}^{n}(b_i)$ and

$\sum_{i=1}^{n}(ka_i)=k\sum_{i=1}^{n}(a_i)$.

Example 1:

Compute the following quantity using the summation formulas:

$$\sum_{i=1}^{10} 2i(i-6i).$$

Solution:

$$\begin{aligned}\sum_{i=1}^{10} 2i(i-6) &= \sum_{i=1}^{10}(2i^2-12i) = 2\sum_{i=1}^{10} i^2 - 12\sum_{i=1}^{10} i \\ &= 2\left(\frac{(10)(10+1)(2\cdot 10+1)}{6}\right) - 12\left(\frac{(10)(11)}{2}\right) \\ &= 770-660 = 110.\end{aligned}$$

Another Look at Upper and Lower Sums

We are now ready to formalize our initial ideas about upper and lower sums.

Let f be a bounded function in a closed interval $[a,b]$ and $P=[x_0,\ldots,x_n]$ the partition of $[a,b]$ into n subintervals.

We can then define the lower and upper sums, respectively, over partition P, by

$$S(P) = \sum_{1}^{n} m_i(x_i-x_{i-1}) = m_1(x_1-x_0)+m_2(x_2-x_1)+\ldots+m_n(x_n-x_{n-1}),$$
$$T(P) = \sum_{1}^{n} M_i(x_i-x_{i-1}) = M_1(x_1-x_0)+M_2(x_2-x_1)+\ldots+M_n(x_n-x_{n-1}).$$

where m_i is the minimum value of f in the interval of length x_i-x_{i-1} and M_i is the maximum value of f in the interval of length x_i-x_{i-1}.

The following example shows how we can use these to find the area.

Example 2:

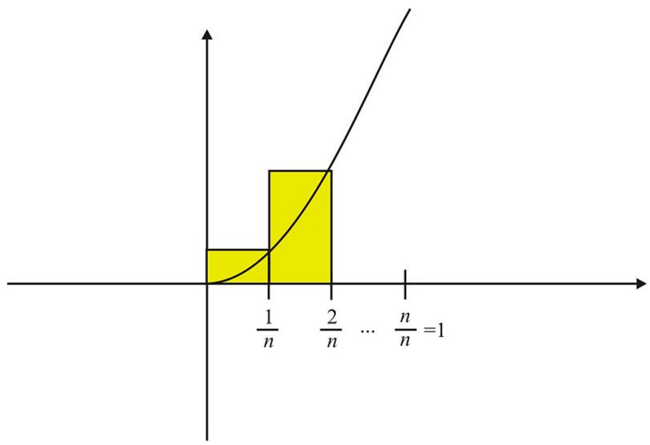
Show that the upper and lower sums for the function $f(x)=x^2$, from $x=0$ to $x=1$, approach the value $A=1/3$.

Solution:

Let P be a partition of n equal sub intervals over $[0,1]$. We will show the result for the upper sums. By our definition we have

$$T(P)=\sum_1^n M_i(x_i-x_{i-1})=M_1(x_1-x_0)+M_2(x_2-x_1)+\ldots+M_n(x_n-x_{n-1}).$$

We note that each rectangle will have width $\frac{1}{n}$, and lengths $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, \left(\frac{n}{n}\right)^2$ as indicated:



$$\begin{aligned}
T(P)&=\sum_1^n M_i(x_i-x_{i-1})=M_1(x_1-x_0)+M_2(x_2-x_1)+\ldots+M_n(x_n-x_{n-1})\\
&=\frac{1}{n}\left(\frac{1}{n}\right)^2+\frac{1}{n}\left(\frac{2}{n}\right)^2+\frac{1}{n}\left(\frac{3}{n}\right)^2+\ldots+\frac{1}{n}\left(\frac{n}{n}\right)^2\\
&=\frac{1}{n}\left(\frac{1}{n}\right)^2(1^2+2^2+3^2+\ldots+n^2)\\
&=\left(\frac{1}{n^3}\right)(1^2+2^2+3^2+\ldots+n^2)=\left(\frac{1}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)=\left(\frac{(n+1)(2n+1)}{6n^2}\right).
\end{aligned}$$

We can re-write this result as:

$$\frac{(n+1)(2n+1)}{6n^2}=\frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)=\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

We observe that as

$$n\to+\infty, \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\to\frac{1}{3}.$$

We now are able to define the area under a curve as a limit.

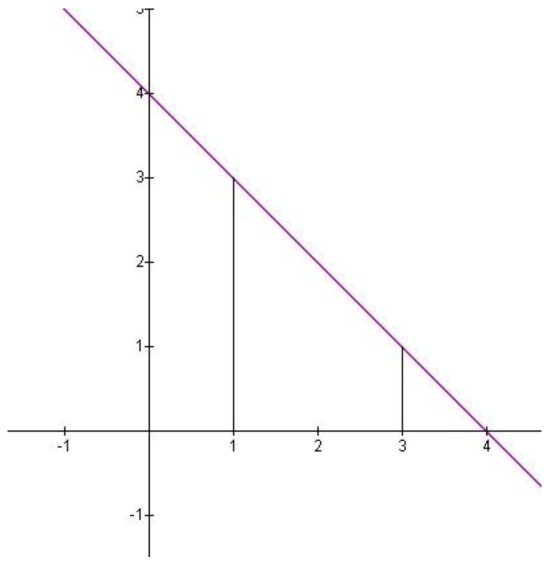
Definition

Let f be a continuous function on a closed interval $[a,b]$. Let P be a partition of n equal sub intervals over $[a,b]$. Then the area under the curve of f is the limit of the upper and lower sums, that is

$$A = \lim_{n \to +\infty} S(P) = \lim_{n \to +\infty} T(P).$$

Example 3:

Use the limit definition of area to find the area under the function $f(x) = 4 - x$ from 1 to $x = 3$.



Solution:

If we partition the interval $[1,3]$ into n equal sub-intervals, then each sub-interval will have length $\frac{3-1}{n} = \frac{2}{n}$ and height $3 - i\triangle x$ as i varies from 1 to n. So we have $\triangle x = \frac{2}{n}$ and

$$\begin{aligned} S(P) &= \sum_1^n (3 - i\triangle x)\triangle x = \sum_1^n (3\triangle x) - \sum_1^n i(\triangle x)^2 \\ &= (3\triangle x)n - \frac{n(n+1)}{2}(\triangle x)^2. \end{aligned}$$

Since $\triangle x = \frac{2}{n}$, we then have by substitution

$(3\triangle x)n - \frac{n(n+1)}{2}(\triangle x)^2 = 6 - \left(2 + \frac{2}{n}\right) = 4 + \frac{2}{n} \to 4$ as $n \to \infty$. Hence the area is $A = 4$.

This example may also be solved with simple geometry. It is left to the reader to confirm that the two methods yield the same area.

Lesson Summary

1. We used sigma notation to evaluate sums of rectangular areas.
2. We found limits of upper and lower sums.
3. We used the limit definition of area to solve problems.

Review Questions

In problems #1–2 , find the summations.

1. $\sum_{i=1}^{10} i(2i-3)$
2. $\sum_{i=1}^{n} (3-i)(2+i)$

In problems #3–5, find $S(P)$ and $T(P)$ under the partition P.

3. $f(x) = 1-x^2, P = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$
4. $f(x) = 2x^2, P = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$
5. $f(x) = \frac{1}{x}, P = \{-4, -3, -2, -1\}$

In problems #6–8, find the area under the curve using the limit definition of area.

6. $f(x) = 3x+5$ from $x = 2$ to $x = 6$.
7. $f(x) = x^2$ from $x = 1$ to $x = 3$.
8. $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 4$.

In problems #9–10, state whether the function is integrable in the given interval. Give a reason for your answer.

9. $f(x) = |x-2|$ on the interval $[1, 4]$
10. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$ on the interval $[0, 1]$

4.4 Definite Integrals

Learning Objectives

- Use Riemann Sums to approximate areas under curves
- Evaluate definite integrals as limits of Riemann Sums

Introduction

In the Lesson The Area Problem we defined the area under a curve in terms of a limit of sums.

$$A = \lim_{n \to +\infty} S(P) = \lim_{n \to +\infty} T(P)$$

where

$$S(P) = \sum_{1}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \ldots + m_n(x_n - x_{n-1}),$$
$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1}),$$

$S(P)$, and $T(P)$ were examples of ***Riemann Sums***. In general, Riemann Sums are of form $\sum_{i=1}^{n} f(x_i^*)\triangle x$ where each x_i^* is the value we use to find the length of the rectangle in the i^{th} sub-interval. For example, we used the maximum function value in each sub-interval to find the upper sums and the minimum function in each sub-interval to find the lower sums. But since the function is continuous, we could have used any points within the sub-intervals to find the limit. Hence we can define the most general situation as follows:

Definition

If f is continuous on $[a,b]$, we divide the interval $[a,b]$ into n sub-intervals of equal width with $\triangle x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these sub-intervals and let $x_1^*, x_2^*, \ldots, x_n^*$ be *any* sample points in these sub-intervals. Then the ***definite integral*** of f from $x = a$ to $x = b$ is

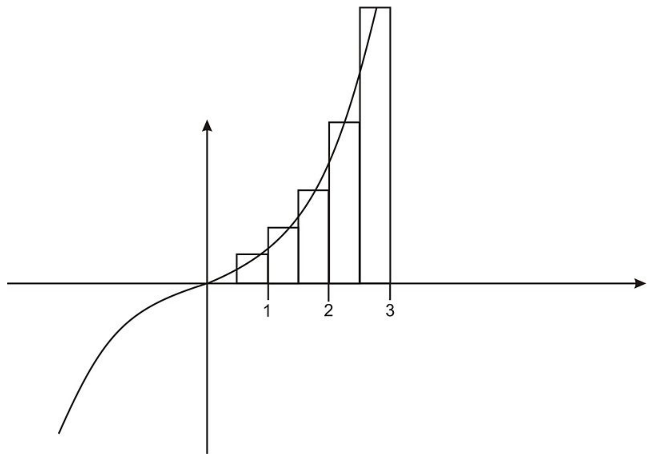
$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\triangle x.$$

Example 1:

Evaluate the Riemann Sum for $f(x) = x^3$ from $x = 0$ to $x = 3$ using $n = 6$ sub-intervals and taking the sample points to be the midpoints of the sub-intervals.

Solution:

If we partition the interval $[0,3]$ into $n=6$ equal sub-intervals, then each sub-interval will have length $\frac{3-0}{6}=\frac{1}{2}$. So we have $\triangle x=\frac{1}{2}$ and



$$\begin{aligned}R_6 &= \sum_1^6 f(x_i^*)\triangle x = f(0.25)\triangle x + f(0.75)\triangle x + f(1.25)\triangle x + f(1.75)\triangle x + f(2.25)\triangle x + f(2.75)\triangle x\\ &= \left(\frac{1}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{27}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{125}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{343}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{729}{64}\right)\left(\frac{1}{2}\right)+\left(\frac{1331}{64}\right)\left(\frac{1}{2}\right)\\ &= \frac{2556}{64} = 39.93.\end{aligned}$$

Now let's compute the definite integral using our definition and also some of our summation formulas.

Example 2:

Use the definition of the definite integral to evaluate $\int_0^3 x^3 dx$.

Solution:

Applying our definition, we need to find

$$\int_0^3 x^3 dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\triangle x.$$

We will use right endpoints to compute the integral. We first need to divide $[0,3]$ into n sub-intervals of length $\triangle x=\frac{3-0}{n}=\frac{3}{n}$. Since we are using right endpoints, $x_0=0, x_1=\frac{3}{n}, x_2=\frac{6}{n},\ldots x_i=\frac{3i}{n}$.

So $\int_0^3 x^3 dx = \lim_{n\to\infty}\sum_{i=1}^n f(\frac{3i}{n})(\frac{3}{n}) = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n(\frac{3i}{n})^3 = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n(\frac{27}{n^3})i^3 = \lim_{n\to\infty}\frac{81}{n^4}\sum_{i=1}^n i^3$.

Recall that $\sum_1^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$. By substitution, we have

$$\int_0^3 x^3 dx = \lim_{n\to\infty} = \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty}\frac{81}{4}\left[1+\frac{1}{n}\right]^2 \to \frac{81}{4} \text{ as } n\to\infty.$$

Hence

$$\int_0^3 x^3 dx = \frac{81}{4}.$$

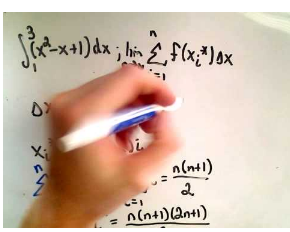
Before we look to try some problems, let's make a couple of observations. First, we will soon not need to rely on the summation formula and Riemann Sums for actual computation of definite integrals. We will develop several computational strategies in order to solve a variety of problems that come up. Second, the idea of definite integrals as approximating the area under a curve can be a bit confusing since we may sometimes get results that do not make sense when interpreted as areas. For example, if we were to compute the definite integral $\int_{-3}^{3} x^3 dx$, then due to the symmetry of $f(x) = x^3$ about the origin, we would find that $\int_{-3}^{3} x^3 dx = 0$. This is because for every sample point x_j^*, we also have $-x_j^*$ is also a sample point with $f(-x_j^*) = -f(x_j^*)$. Hence, it is more accurate to say that $\int_{-3}^{3} x^3 dx$ gives us the ***net area*** between $x = -3$ and $x = 3$. If we wanted the ***total area*** bounded by the graph and the $x-$axis, then we would compute $2\int_0^3 x^3 dx = \frac{81}{2}$.

Lesson Summary

1. We used Riemann Sums to approximate areas under curves.
2. We evaluated definite integrals as limits of Riemann Sums.

Multimedia Link

For video presentations on calculating definite integrals using Riemann Sums **(13.0)**, see Riemann Sums, Part 1 (6:15)



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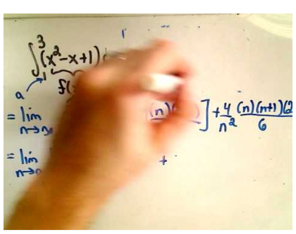
and Riemann Sums, Part 2 (8:31).



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The following applet lets you explore Riemann Sums of any function. You can change the bounds and the number of partitions. Follow the examples given on the page, and then use the applet to explore on your own. Riemann Sums Applet . Note: On this page the author uses Left- and Right- hand sums. These are similar to the sums $S(P)$ and $T(P)$ that you have learned, particularly in the case of an increasing (or decreasing) function. Left-hand

and Right-hand sums are frequently used in calculations of numerical integrals because it is easy to find the left and right endpoints of each interval, and much more difficult to find the max/min of the function on each interval. The difference is not always important from a numerical approximation standpoint; as you increase the number of partitions, you should see the Left-hand and Right-hand sums converging to the same value. Try this in the applet to see for yourself.

Review Questions

In problems #1–7 , use Riemann Sums to approximate the areas under the curves.

1. Consider $f(x) = 2-x$ from $x = 0$ to $x = 2$. Use Riemann Sums with four subintervals of equal lengths. Choose the midpoints of each subinterval as the sample points.
2. Repeat problem #1 using geometry to calculate the exact area of the region under the graph of $f(x) = 2-x$ from $x = 0$ to $x = 2$. (Hint: Sketch a graph of the region and see if you can compute its area using area measurement formulas from geometry.)
3. Repeat problem #1 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x) = 2-x$ from $x = 0$ to $x = 2$.
4. $f(x) = x^2 - x$ from $x = 1$ to $x = 4$. Use Riemann Sums with five subintervals of equal lengths. Choose the left endpoint of each subinterval as the sample points.
5. Repeat problem #4 using the definition of the definite intergal to calculate the exact area of the region under the graph of $f(x) = x^2 - x$ from $x = 1$ to $x = 4$.
6. Consider $f(x) = 3x^2$. Compute the Riemann Sum of f on $[0,1]$ under each of the following situations. In each case, use the right endpoint as the sample points.
   a. Two sub-intervals of equal length.
   b. Five sub-intervals of equal length.
   c. Ten sub-intervals of equal length.
   d. Based on your answers above, try to guess the exact area under the graph of f on $[0,1]$.
7. Consider $f(x) = e^x$. Compute the Riemann Sum of f on $[0,1]$ under each of the following situations. In each case, use the right endpoint as the sample points.
   a. Two sub-intervals of equal length.
   b. Five sub-intervals of equal length.
   c. Ten sub-intervals of equal length.
   d. Based on your answers above, try to guess the exact area under the graph of f on $[0,1]$.
8. Find the net area under the graph of $f(x) = x^3 - x$; $x = -1$ to $x = 1$. (Hint: Sketch the graph and check for symmetry.)
9. Find the total area bounded by the graph of $f(x) = x^3 - x$ and the $x-$axis, from to $x = -1$ to $x = 1$.
10. Use your knowledge of geometry to evaluate the following definite integral: $\int_0^3 \sqrt{9-x^2}dx$ (Hint: set $y = \sqrt{9-x^2}$ and square both sides to see if you can recognize the region from geometry.)

4.5 Evaluating Definite Integrals

Learning Objectives

- Use antiderivatives to evaluate definite integrals
- Use the Mean Value Theorem for integrals to solve problems
- Use general rules of integrals to solve problems

Introduction

In the Lesson on Definite Integrals, we evaluated definite integrals using the limit definition. This process was long and tedious. In this lesson we will learn some practical ways to evaluate definite integrals. We begin with a theorem that provides an easier method for evaluating definite integrals. Newton discovered this method that uses antiderivatives to calculate definite integrals.

Theorem 4.1:

If f is continuous on the closed interval $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any antiderivative of f.

We sometimes use the following shorthand notation to indicate $\int_a^b f(x)dx = F(b) - F(a)$:

$$\int_a^b f(x)dx = F(x)]_a^b.$$

The proof of this theorem is included at the end of this lesson. Theorem 4.1 is usually stated as a part of the Fundamental Theorem of Calculus, a theorem that we will present in the Lesson on the Fundamental Theorem of Calculus. For now, the result provides a useful and efficient way to compute definite integrals. We need only find an antiderivative of the given function in order to compute its integral over the closed interval. It also gives us a result with which we can now state and prove a version of the Mean Value Theorem for integrals. But first let's look at a couple of examples.

Example 1:

Compute the following definite integral:

$$\int_0^3 x^3 dx.$$

Solution:

Using the limit definition we found that $\int_0^3 x^3 dx = \frac{81}{4}$. We now can verify this using the theorem as follows:

We first note that $x^4/4$ is an antiderivative of $f(x) = x^3$. Hence we have

$$\int_0^3 x^3 dx = \frac{x^4}{4}\Big]_0^3 = \frac{81}{4} - \frac{0}{4} = \frac{81}{4}.$$

We conclude the lesson by stating the rules for definite integrals, most of which parallel the rules we stated for the general indefinite integrals.

$$\int_a^a f(x)dx = 0$$
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx$$
$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \text{ where } a < c < b.$$

Given these rules together with Theorem 4.1, we will be able to solve a great variety of definite integrals.

Example 2:

Compute $\int_{-2}^2 (x - \sqrt{x})dx$.

Solution:

$$\int_1^4 (x - \sqrt{x})dx = \int_1^4 xdx - \int_1^4 \sqrt{x}dx = \frac{x^2}{2}\Big]_1^4 - \frac{2}{3}x^{\frac{3}{2}}\Big]_1^4 = \left(8 - \frac{1}{2}\right) - \frac{2}{3}(8-1) = \frac{15}{2} - \frac{14}{3} = \frac{17}{6}.$$

Example 3:

Compute $\int_0^{\frac{\pi}{2}} (x + \cos x)dx$.

Solution:

$$\int_0^{\frac{\pi}{2}} (x + \cos x)dx = \int_0^{\frac{\pi}{2}} (x)dx = \int_0^{\frac{\pi}{2}} (\cos x)dx = \frac{x^2}{2}\Big|_0^{\frac{\pi}{2}} + \frac{\sin x}{1}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} + 1 = \frac{\pi^2 + 8}{8}.$$

Lesson Summary

1. We used antiderivatives to evaluate definite integrals.
2. We used the Mean Value Theorem for integrals to solve problems.
3. We used general rules of integrals to solve problems.

Proof of Theorem 4.1

We first need to divide $[a,b]$ into n sub-intervals of length $\triangle x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these sub-intervals.

Let F be any antiderivative of f.

Consider $F(b) - F(a) = F(x_n) - F(x_0)$.

We will now employ a method that will express the right side of this equation as a Riemann Sum. In particular,

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) - \ldots + F(x_1) - F(x_0) \\ &= \sum_1^n [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Note that F is continuous. Hence, by the Mean Value Theorem, there exist $c_i \in [x_i - x_{i-1}, x_i]$

such that $F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\triangle x$.

Hence

$$F(b) - F(a) = \sum_1^n F'(c_i)(x_i - x_{i-1}) = \sum_1^n f(c_i)\triangle x.$$

Taking the limit of each side as $n \to \infty$ we have

$$\lim_{n \to \infty} [F(b) - F(a)] = \lim_{n \to \infty} \sum_1^n f(c_i)\triangle x.$$

We note that the left side is a constant and the right side is our definition for $\int_a^b f(x)dx$.

Hence

$$F(b) - F(a) = \lim_{n \to \infty} \sum_1^n f(c_i)\triangle x = \int_a^b f(x)dx.$$

Proof of Theorem 4.2

Let $F(x) = \int_a^x f(x)dx$.

By the Mean Value Theorem for derivatives, there exists $c \in [a,b]$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

From Theorem 4.1 we have that F is an antiderivative of f. Hence, $F'(x) = f(x)$ and in particular, $F'(c) = f(c)$. Hence, by substitution we have

$$f(c) = \frac{F(b) - F(a)}{b - a}.$$

Note that $F(a) = \int_a^a f(x)dx = 0$. Hence we have

$$f(c) = \frac{F(b) - 0}{b - a} = \frac{F(b)}{b - a},$$

and by our definition of $F(x)$ we have

$$f(c) = \frac{1}{b-a} F(b) = \frac{1}{b-a} \int_a^b f(x)dx.$$

This theorem allows us to find for positive functions a rectangle that has base $[a, b]$ and height $f(c)$ such that the area of the rectangle is the same as the area given by $\int_a^b f(x)dx$. In other words, $f(c)$ is the average function value over $[a, b]$.

Review Questions

In problems #1–8, use antiderivatives to compute the definite integral.

1. $\int_4^9 (\frac{3}{\sqrt{x}})dx$
2. $\int_0^1 (t - t^2)dt$
3. $\int_2^5 (\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{2}})dx$
4. $\int_0^1 4(x^2 - 1)(x^2 + 1)dx$
5. $\int_2^8 (\frac{4}{x} + x^2 + x)dx$
6. $\int_2^4 (e^{3x})dx$
7. $\int_1^4 \frac{2}{x+3} dx$
8. Find the average value of $f(x) = \sqrt{x}$ over $[1, 9]$.
9. If f is continuous and $\int_1^4 f(x)dx = 9$, show that f takes on the value 3 at least once on the interval $[1, 4]$.
10. Your friend states that there is no area under the curve of $f(x) = \sin x$ on $[0, 2\pi]$ since he computed $\int_0^{2\pi} \sin x dx = 0$. Is he correct? Explain your answer.

4.6 The Fundamental Theorem of Calculus

Learning Objectives

- Use the Fundamental Theorem of Calculus to evaluate definite integrals

Introduction

In the Lesson on Evaluating Definite Integrals, we evaluated definite integrals using antiderivatives. This process was much more efficient than using the limit definition. In this lesson we will state the Fundamental Theorem of Calculus and continue to work on methods for computing definite integrals.

Fundamental Theorem of Calculus:

Let f be continuous on the closed interval $[a,b]$.

1. If function F is defined by $F(x) = \int_a^x f(t)dt$, on $[a,b]$, then $F'(x) = f(x)$ on $[a,b]$.
2. If g is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(t)dt = g(b) - g(a).$$

We first note that we have already proven part 2 as Theorem 4.1. The proof of part 1 appears at the end of this lesson.

Think about this Theorem. Two of the major unsolved problems in science and mathematics turned out to be solved by calculus which was invented in the seventeenth century. These are the ancient problems:

1. Find the areas defined by curves, such as circles or parabolas.
2. Determine an instantaneous rate of change or the slope of a curve at a point.

With the discovery of calculus, science and mathematics took huge leaps, and we can trace the advances of the space age directly to this Theorem.

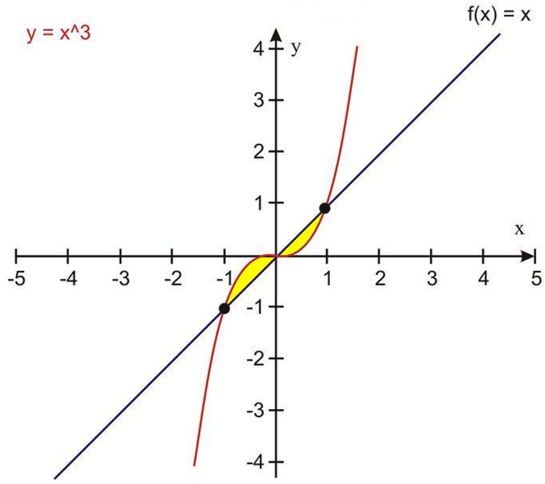
Let's continue to develop our strategies for computing definite integrals. We will illustrate how to solve the problem of finding the area bounded by two or more curves.

Example 1:

Find the area between the curves of $f(x) = x$ and $g(x) = x^3$. for $-1 \le x \le 1$.

Solution:

We first observe that there are no limits of integration explicitly stated here. Hence we need to find the limits by analyzing the graph of the functions.



We observe that the regions of interest are in the first and third quadrants from $x = -1$ to $x = 1$. We also observe the symmetry of the graphs about the origin. From this we see that the total area enclosed is

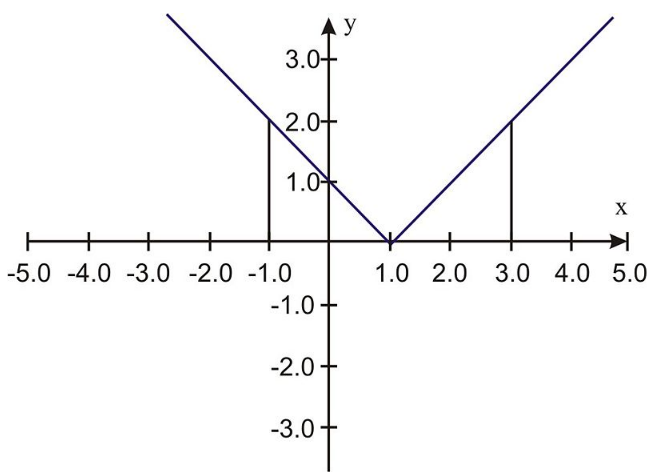
$$2\int_0^1 (x-x^3)dx = 2\left[\int_0^1 xdx - \int_0^1 x^3 dx\right] = 2\left[\frac{x^2}{2}\bigg|_0^1 - \frac{x^4}{4}\bigg|_0^1 =\right] = 2\left[\frac{1}{2}-\frac{1}{4}\right] = 2\left[\frac{1}{4}\right] = \frac{1}{2}.$$

Example 2:

Find the area between the curves of $f(x) = |x-1|$ and the $x-$axis from $x = -1$ to $x = 3$.

Solution:

We observe from the graph that we will have to divide the interval $[-1,3]$ into subintervals $[-1,1]$ and $[1,3]$.



Hence the area is given by

$$\int_{-1}^{1}(-x+1)dx + \int_1^3 (x-1)dx = \left(-\frac{x^2}{2}+x\right)\bigg|_{-1}^{+1} + \left(\frac{x^2}{2}-x\right)\bigg|_{+1}^{+3} = 2+2 = 4.$$

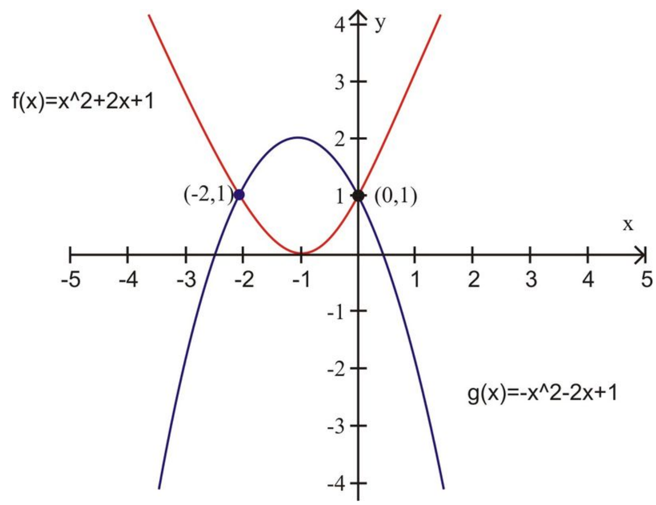
Example 3:

Find the area enclosed by the curves of $f(x) = x^2+2x+1$ and

$$g(x) = -x^2 - 2x + 1.$$

Solution:

The graph indicates the area we need to focus on.



$$\int_{-2}^{0}(-x^2 - 2x + 1)dx - \int_{-2}^{0}(x^2 + 2x + 1)dx = \left(-\frac{x^3}{3} - x^2 + x\right)\Big|_{-2}^{0} - \left(\frac{x^3}{3} + x^2 + x\right)\Big|_{-2}^{0} = \frac{8}{3}.$$

Before providing another example, let's look back at the first part of the Fundamental Theorem. If function F is defined by $F(x) = \int_a^x f(t)dt$, on $[a,b]$ then $F'(x) = f(x)$ on $[a,b]$. Observe that if we differentiate the integral with respect to x, we have

$$\frac{d}{dx}\int_a^x f(t)dt = F'(x) = f(x).$$

This fact enables us to compute derivatives of integrals as in the following example.

Example 4:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x) = \int_0^x (1 + \sqrt[3]{t})dt.$$

Solution:

While we could easily integrate the right side and then differentiate, the Fundamental Theorem enables us to find the answer very routinely.

$$g'(x) = \frac{d}{dx}\int_0^x (1 + \sqrt[3]{t})dt = 1 + \sqrt[3]{x}.$$

This application of the Fundamental Theorem becomes more important as we encounter functions that may be more difficult to integrate such as the following example.

Example 5:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x) = \int_2^x (t^2 \cos t) dt.$$

Solution:

In this example, the integral is more difficult to evaluate. The Fundamental Theorem enables us to find the answer routinely.

$$g'(x) = \frac{d}{dx} \int_2^x (t^2 \cos t) dt = x^2 \cos x.$$

Lesson Summary

1. We used the Fundamental Theorem of Calculus to evaluate definite integrals.

Fundamental Theorem of Calculus

Let f be continuous on the closed interval $[a,b]$.

1. If function F is defined by $F(x) = \int_a^x f(t)dt$, on $[a,b]$, then $F'(x) = f(x)$, on $[a,b]$.
2. If g is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(t)dt = g(b) - g(a).$$

We first note that we have already proven part 2 as Theorem 4.1.

Proof of Part 1.'

1. Consider $F(x) = \int_a^x f(t)dt$, on $[a,b]$.

2. $x, c \in [a,b], c < x$.

Then $\int_a^x f(t)dt = \int_a^c f(t)dt + \int_c^x f(t)dt$ by our rules for definite integrals.

3. Then $\int_a^x f(t)dt - \int_a^c f(t)dt = \int_c^x f(t)dt$. Hence $F(x) - F(c) = \int_c^x f(t)dt$.

4. Since f is continuous on $[a,b]$ and $x, c \in [a,b], c < x$ then we can select $u, v \in [c, v]$ such that $f(u)$ is the minimum value of and $f(v)$ is the maximum value of f in $[c,x]$. Then we can consider $f(u)(x-c)$ as a lower sum and $f(v)(x-c)$ as an upper sum of f from c to x. Hence

5. $f(u)(x-c) \le \int_c^x f(t)dt \le f(v)(x-c)$.

6. By substitution, we have:

$$f(u)(x-c) \le F(x) - F(c) \le f(v)(x-c).$$

7. By division, we have

$$f(u) \leq \frac{F(x)-F(c)}{x-c} \leq f(v).$$

8. When x is close to c, then both $f(u)$ and $f(v)$ are close to $f(c)$ by the continuity of f

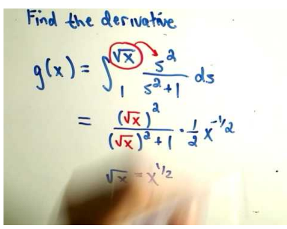
9. Hence $\lim_{x \to c^+} \frac{F(x)-F(c)}{x-c} = f(c)$. Similarly, if $x < c$, then $\lim_{x \to c^-} \frac{F(x)-F(c)}{x-c} = f(c)$. Hence, $\lim_{x \to c} \frac{F(x)-F(c)}{x-c} = f(c)$.

10. By the definition of the derivative, we have that

$F'(c) = \lim_{x \to c} \frac{F(x)-F(c)}{x-c} = f(c)$ for every $c \in [a,b]$. Thus, F is an antiderivative of f on $[a,b]$.

Multimedia Link

For a video presentation of the Fundamental Theorem of Calculus **(15.0)**, see Fundamental Theorem of Calculus, Part 1 (9:26).



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Review Questions

In problems #1–4, sketch the graph of the function $f(x)$ in the interval $[a,b]$. Then use the Fundamental Theorem of Calculus to find the area of the region bounded by the graph and the $x-$axis.

1. $f(x) = 2x+3, [0,4]$
2. $f(x) = e^x, [0,2]$
3. $f(x) = x^2+x, [1,3]$
4. $f(x) = x^2-x, [0,2]$

(Hint: Examine the graph of the function and divide the interval accordingly.)

In problems #5–7 use antiderivatives to compute the definite integral.

5. $\int_{-1}^{+1} |x| dx$
6. $\int_0^3 |x^3-2| dx$

(Hint: Examine the graph of the function and divide the interval accordingly.)

7. $\int_{-2}^{+4} [|x-1|+|x+1|] dx$

(Hint: Examine the graph of the function and divide the interval accordingly.)

In problems #8–10, find the area between the graphs of the functions.

8. $f(x) = \sqrt{x}, g(x) = x, [0,2]$
9. $f(x) = x^2, g(x) = 4, [0,2]$
10. $f(x) = x^2 + 1, g(x) = 3 - x, [0,3]$

4.7 Integration by Substitution

Learning Objectives

- Integrate composite functions
- Use change of variables to evaluate definite integrals
- Use substitution to compute definite integrals

Introduction

In this lesson we will expand our methods for evaluating definite integrals. We first look at a couple of situations where finding antiderivatives requires special methods. These involve finding antiderivatives of composite functions and finding antiderivatives of products of functions.

Antiderivatives of Composites

Suppose we needed to compute the following integral:

$$\int 3x^2\sqrt{1+x^3}dx.$$

Our rules of integration are of no help here. We note that the integrand is of the form $f(g(x)) * g'(x)$ where $g(x) = 1+x^3$ and $f(x) = \sqrt{x}$.

Since we are looking for an antiderivative F of f, and we know that $F' = f$, we can re-write our integral as

$$\int \sqrt{1+x^3} \cdot 3x^2 dx = \frac{2}{3}(\sqrt{1+x^3})^{\frac{3}{2}} + C.$$

In practice, we use the following substitution scheme to verify that we can integrate in this way:

1. Let $u = 1+x^3$.
2. Differentiate both sides so $du = 3x^2 dx$.
3. Change the original integral in x to an integral in u:
$\int \sqrt{1+x^3} \cdot 3x^2 dx = \int \sqrt{u} du$, where $u = 1+x^3$ and $du = 3x^2 dx$.
4. Integrate with respect to u:

$$\int \sqrt{u} du = \int u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + C.$$

5. Change the answer back to x:

$$\int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(\sqrt{1+x^3})^{\frac{3}{2}} + C.$$

While this method of substitution is a very powerful method for solving a variety of problems, we will find that we sometimes will need to modify the method slightly to address problems, as in the following example.

Example 1:

Compute the following indefinite integral:

$$\int x^2 e^{x^3} dx.$$

Solution:

We note that the derivative of x^3 is $3x^2$; hence, the current problem is not of the form $\int F'(g(x)) \cdot g'(x)dx$. But we notice that the derivative is off only by a constant of 3 and we know that constants are easy to deal with when differentiating and integrating. Hence

Let $u = x^3$.

Then $du = 3x^2 dx$.

Then $\frac{1}{3}du = x^2 dx$. and we are ready to change the original integral from x to an integral in u and integrate:

$$\int x^2 e^{x^3} dx = \int e^u \left(\frac{1}{3}du\right) = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C.$$

Changing back to x, we have

$$\int x^2 e^{x^3} dx = \frac{1}{3}e^{x^3} + C.$$

We can also use this substitution method to evaluate definite integrals. If we attach limits of integration to our first example, we could have a problem such as

$$\int_1^4 \sqrt{1+x^3} \cdot 3x^2 dx.$$

The method still works. However, we have a choice to make once we are ready to use the Fundamental Theorem to evaluate the integral.

Recall that we found that $\int \sqrt{1+x^3} \cdot 3x^2 dx = \int \sqrt{u}du$ for the indefinite integral. At this point, we could evaluate the integral by changing the answer back to x or we could evaluate the integral in u. But we need to be careful. Since the original limits of integration were in x, we need to change the limits of integration for the equivalent integral in u. Hence,

$\int_1^4 \sqrt{1+x^3} \cdot 3x^2 dx = \int_{u=2}^{65} \sqrt{udu}$, where $u = 1+x^3$

$$\int_1^4 \sqrt{1+x^3} \cdot 3x^2 dx = \int_{u=2}^{65} \sqrt{udu} = \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=2}^{u=65} = \frac{2}{3}(\sqrt{65^3} - \sqrt{8}).$$

Integrating Products of Functions

We are not able to state a rule for integrating products of functions, $\int f(x)g(x)dx$ but we can get a relationship that is almost as effective. Recall how we differentiated a product of functions:

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x).$$

So by integrating both sides we get

$\int [f(x)g'(x) + g(x)f'(x)]dx = f(x)g(x)$, or

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x).$$

In order to remember the formula, we usually write it as

$$\int udv = uv - \int vdu.$$

We refer to this method as integration by parts. The following example illustrates its use.

Example 2:

Use integration by parts method to compute

$$\int xe^x dx.$$

Solution:

We note that our other substitution method is not applicable here. But our integration by parts method will enable us to reduce the integral down to one that we can easily evaluate.

Let $u = x$ and $dv = e^x dx$ then $du = dx$ and $v = e^x$

By substitution, we have

$$\int xe^x dx = xe^x - \int e^x dx.$$

We can easily evaluate the integral and have

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C.$$

And should we wish to evaluate definite integrals, we need only to apply the Fundamental Theorem to the antiderivative.

Lesson Summary

1. We integrated composite functions.
2. We used change of variables to evaluate definite integrals.
3. We used substitution to compute definite integrals.

Review Questions

Compute the integrals in problems #1–10.

1. $\int x \ln x dx$
2. $\int_1^3 \sqrt{x} \ln x dx$
3. $\int \frac{x}{\sqrt{2x+1}} dx$
4. $\int_0^1 x^3 \sqrt{1-x^2} dx$
5. $\int x \cos x dx$
6. $\int_0^1 x^2 \sqrt{x^3+9} dx$
7. $\int \left(\frac{1}{x^2} \cdot e^{\frac{1}{x}} \right) dx$
8. $\int x^3 e^{x^2} dx$
9. $\int \frac{\ln x}{x^{\frac{5}{2}}} dx$
10. $\int_1^e \frac{1}{x} dx$

4.8 Numerical Integration

Learning Objectives

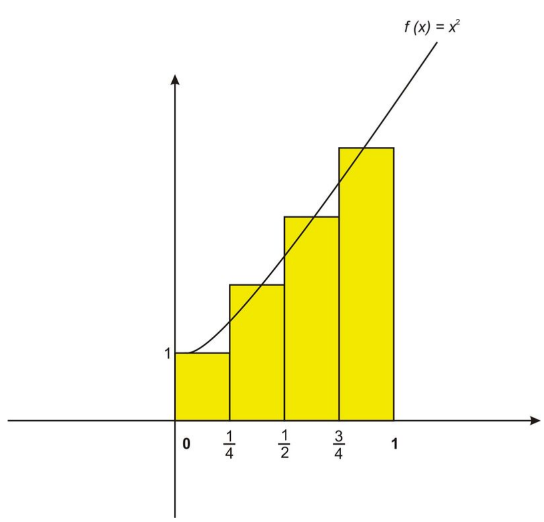
- Use the Trapezoidal Rule to solve problems
- Estimate errors for the Trapezoidal Rule
- Use Simpson's Rule to solve problems
- Estimate Errors for Simpson's Rule

Introduction

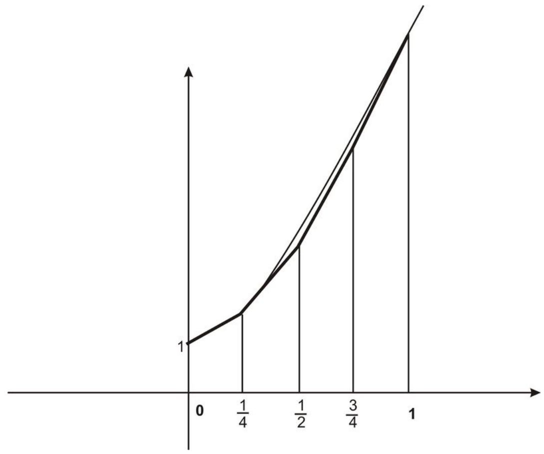
Recall that we used different ways to approximate the value of integrals. These included Riemann Sums using left and right endpoints, as well as midpoints for finding the length of each rectangular tile. In this lesson we will learn two other methods for approximating integrals. The first of these, the Trapezoidal Rule, uses areas of trapezoidal tiles to approximate the integral. The second method, Simpson's Rule, uses parabolas to make the approximation.

Trapezoidal Rule

Let's recall how we would use the midpoint rule with $n = 4$ rectangles to approximate the area under the graph of $f(x) = x^2 + 1$ from $x = 0$ to $x = 1$.



If instead of using the midpoint value within each sub-interval to find the length of the corresponding rectangle, we could have instead formed trapezoids by joining the maximum and minimum values of the function within each sub-interval:



The area of a trapezoid is $A = \frac{h(b_1+b_2)}{2}$, where b_1 and b_2 are the lengths of the parallel sides and h is the height. In our trapezoids the height is $\triangle x$ and b_1 and b_2 are the values of the function. Therefore in finding the areas of the trapezoids we actually average the left and right endpoints of each sub-interval. Therefore a typical trapezoid would have the area

$$A = \frac{\triangle x}{2}\left(f(x_{i-1}) + f(x_i)\right).$$

To approximate $\int_a^b f(x)dx$ with n of these trapezoids, we have

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{1}{2}\left[\sum_{i=1}^n f(x_{i-1})\triangle x + \sum_{i=1}^n f(x_i)\triangle x\right]\\ &= \frac{\triangle x}{2}\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n)\right]\\ &= \frac{\triangle x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right], \triangle x = \frac{b-a}{n}.\end{aligned}$$

Example 1:

Use the Trapezoidal Rule to approximate $\int_0^3 x^2dx$ with $n = 6$.

Solution:

We find $\triangle x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$.

$$\begin{aligned}\int_0^3 x^2dx &\approx \frac{1}{4}\left[f(0) + 2f(\tfrac{1}{2}) + 2f(1) + 2f(\tfrac{3}{2}) + 2f(2) + 2f(\tfrac{5}{2}) + f(3)\right]\\ &= \frac{1}{4}\left[0 + (2\cdot\tfrac{1}{4}) + (2\cdot 1) + (2\cdot\tfrac{9}{4}) + (2\cdot 4) + (2\cdot\tfrac{25}{4}) + 9\right]\\ &= \frac{1}{4}\left[\tfrac{73}{2}\right] = \frac{73}{8} = 9.125.\end{aligned}$$

Of course, this estimate is not nearly as accurate as we would like. For functions such as $f(x) = x^2$, we can easily find an antiderivative with which we can apply the Fundamental Theorem that $\int_0^3 x^2dx = \left.\frac{x^3}{3}\right]_0^3 = 9$. But it is not always easy to find an antiderivative. Indeed, for many integrals it is impossible to find an antiderivative. Another issue concerns the questions about the accuracy of the approximation. In particular, how large should we take n so that the Trapezoidal Estimate for $\int_0^3 x^2 dx$ is accurate to within a given value, say 0.001? As with our Linear Approximations in the Lesson on Approximation Errors, we can state a method that ensures our approximation to be within a specified value.

Error Estimates for Simpson's Rule

We would like to have confidence in the approximations we make. Hence we can choose n to ensure that the errors are within acceptable boundaries. The following method illustrates how we can choose a sufficiently large n.

Suppose $|f''(x)| \leq k$ for $a \leq x \leq b$. Then the error estimate is given by

$$|Error_{Trapezoidal}| \leq \frac{k(b-a)^3}{12n^2}.$$

Example 2:

Find n so that the Trapezoidal Estimate for $\int_0^3 x^2 dx$ is accurate to 0.001.

Solution:

We need to find n such that $|Error_{Trapezoidal}| \leq 0.001$. We start by noting that $|f''(x)| = 2$ for $0 \leq x \leq 3$. Hence we can take $k = 2$ to find our error bound.

$$|Error_{Trapezoidal}| \leq \frac{2(3-0)^3}{12n^2} = \frac{54}{12n^2}.$$

We need to solve the following inequality for n:

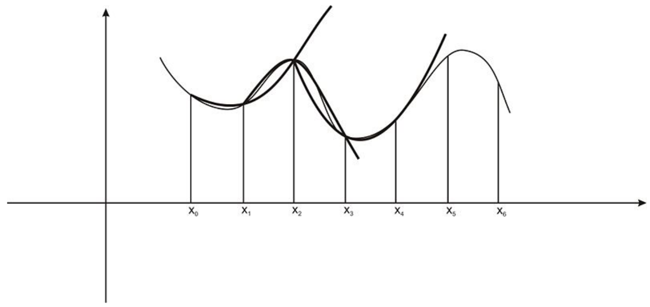
$$\begin{aligned} \frac{54}{12n^2} &< 0.001, \\ n^2 &> \frac{54}{12(0.001)}, \\ n &> \sqrt{\frac{54}{12(0.001)}} \approx 67.08. \end{aligned}$$

Hence we must take $n = 68$ to achieve the desired accuracy.

From the last example, we see one of the weaknesses of the Trapezoidal Rule—it is not very accurate for functions where straight line segments (and trapezoid tiles) do not lead to a good estimate of area. It is reasonable to think that other methods of approximating curves might be more applicable for some functions. ***Simpson's Rule*** is a method that uses parabolas to approximate the curve.

Simpson's Rule:

As was true with the Trapezoidal Rule, we divide the interval $[a,b]$ into n sub-intervals of length $\triangle x = \frac{b-a}{n}$. We then construct parabolas through each group of three consecutive points on the graph. The graph below shows this process for the first three such parabolas for the case of $n = 6$ sub-intervals. You can see that every interval except the first and last contains two estimates, one too high and one too low, so the resulting estimate will be more accurate.



Using parabolas in this way produces the following estimate of the area from Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{\triangle x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right].$$

We note that it has a similar appearance to the Trapezoidal Rule. However, there is one distinction we need to note. The process of using three consecutive x_i to approximate parabolas will require that we assume that n must always be an even number.

Error Estimates for the Trapezoidal Rule

As with the Trapezoidal Rule, we have a formula that suggests how we can choose n to ensure that the errors are within acceptable boundaries. The following method illustrates how we can choose a sufficiently large n.

Suppose $|f^4(x)| \le k$ for $a \le x \le b$. Then the error estimate is given by

$$|Error_{simpson}| \le \frac{k(b-a)^5}{180n^4}.$$

Example 3:

a. Use Simpson's Rule to approximate $\int_1^4 \frac{1}{x}dx$ with $n=6$.

Solution:

We find $\triangle x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{1}{2}$.

$$\begin{aligned}\int_1^4 \frac{1}{x}dx &\approx \frac{1}{6}\left[f(1)+4f(\tfrac{3}{2})+2f(2)+4f(\tfrac{5}{2})+2f(3)+4f(\tfrac{7}{2})+f(4)\right]\\ &= \frac{1}{6}\left[1+(4\cdot\tfrac{2}{3})+(2\cdot\tfrac{1}{2})+(4\cdot\tfrac{2}{5})+(2\cdot\tfrac{1}{3})+(4\cdot\tfrac{2}{7})+\tfrac{1}{4}\right]\\ &= \frac{1}{6}\left[\tfrac{3517}{420}\right] = 1.3956.\end{aligned}$$

This turns out to be a pretty good estimate, since we know that

$$\int_1^4 \frac{1}{x}dx = \ln x\Big]_1^4 = \ln(4)-\ln(1) = 1.3863.$$

Therefore the error is less than 0.01.

b. Find n so that the Simpson Rule Estimate for $\int_1^4 \frac{1}{x}dx$ is accurate to 0.001.

Solution:

We need to find n such that $|Error_{simpson}| \leq 0.001$. We start by noting that $\left|f^4(x)\right| = \left|\frac{24}{x^5}\right|$ for $1 \leq x \leq 4$. Hence we can take $K = 24$ to find our error bound:

$$|Error_{simpson}| \leq \frac{24(4-1)^5}{180n^4} = \frac{5832}{180n^4}.$$

Hence we need to solve the following inequality for n:

$$\frac{5832}{180n^4} < 0.001.$$

We find that

$$n^4 > \frac{5832}{180(0.001)},$$

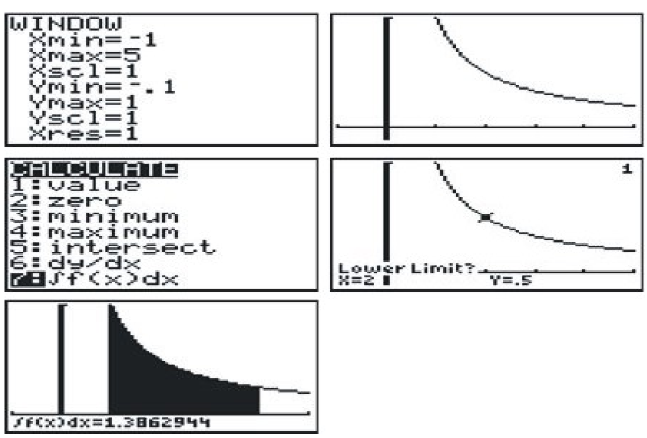
$$n > \sqrt[4]{\frac{5832}{180(0.001)}} \approx 13.42.$$

Hence we must take $n = 14$ to achieve the desired accuracy.

Technology Note: Estimating a Definite Integral with a TI-83/84 Calculator

We will estimate the value of $\int_1^4 \frac{1}{x} dx$.

1. Graph the function $f(x) = \frac{1}{x}$ with the [WINDOW] setting shown below.
2. The graph is shown in the second screen.
3. Press **2nd [CALC]** and choose option **7** (see menu below)
4. When the fourth screen appears, press **[1] [ENTER]** then **[4] [ENTER]** to enter the lower and upper limits.
5. The final screen gives the estimate, which is accurate to 7 decimal places.



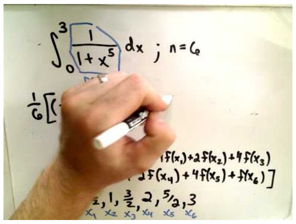
Lesson Summary

1. We used the Trapezoidal Rule to solve problems.

2. We estimated errors for the Trapezoidal Rule.
3. We used Simpson's Rule to solve problems.
4. We estimated Errors for Simpson's Rule.

Multimedia Links

For video presentations of Simpson's Rule **(21.0)**, see Simpson's Rule, Approximate Integration (7:21)



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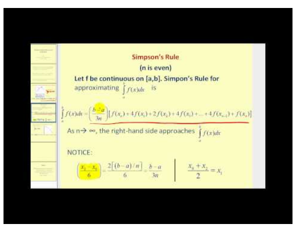
and Math Video Tutorials by James Sousa, Simpson's Rule of Numerical Integration (8:48).



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Review Questions

1. Use the Trapezoidal Rule to approximate $\int_0^1 x^2 e^{-x} dx$ with $n = 8$.
2. Use the Trapezoidal Rule to approximate $\int_1^4 \ln \sqrt{x} dx$ with $n = 6$.
3. Use the Trapezoidal Rule to approximate $\int_0^1 \sqrt{1 + x^4} dx$ with $n = 4$.
4. Use the Trapezoidal Rule to approximate $\int_1^3 \frac{1}{x} dx$ with $n = 8$.
5. How large should you take n so that the Trapezoidal Estimate for $\int_1^3 \frac{1}{x} dx$ is accurate to within 0.001?
6. Use Simpson's Rule to approximate $\int_0^1 x^2 e^{-x} dx$ with $n = 8$.
7. Use Simpson's Rule to approximate $\int_1^4 \sqrt{x} \ln x dx$ with $n = 6$.
8. Use Simpson's Rule to approximate $\int_0^2 \frac{1}{\sqrt{x^4 + 1}} dx$ with $n = 6$.
9. Use Simpson's Rule to approximate $\int_0^1 \sqrt{1 + x^4} dx$ with $n = 4$.
10. How large should you take n so that the Simpson Estimate for $\int_0^2 e dx$ is accurate to within 0.00001?

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9729 .

CHAPTER 5 Applications of Definite Integrals

Chapter Outline

In this chapter, we will explore some of the many applications of the definite integral by using it to calculate areas between two curves, volumes, length of curves, and several other applications from real life such as calculating the work done by a force, the pressure a liquid exerts on an object, and basic statistical concepts.

5.1 Area Between Two Curves

Learning Objectives

A student will be able to:

- Compute the area between two curves with respect to the $x-$ and $y-$axes.

In the last chapter, we introduced the definite integral to find the area between a curve and the $x-$ axis over an interval $[a,b]$. In this lesson, we will show how to calculate the area between two curves.

Consider the region bounded by the graphs f and g between $x = a$ and $x = b$, as shown in the figures below. If the two graphs lie above the $x-$axis, we can interpret the area that is sandwiched between them as the area under the graph of g subtracted from the area under the graph f.

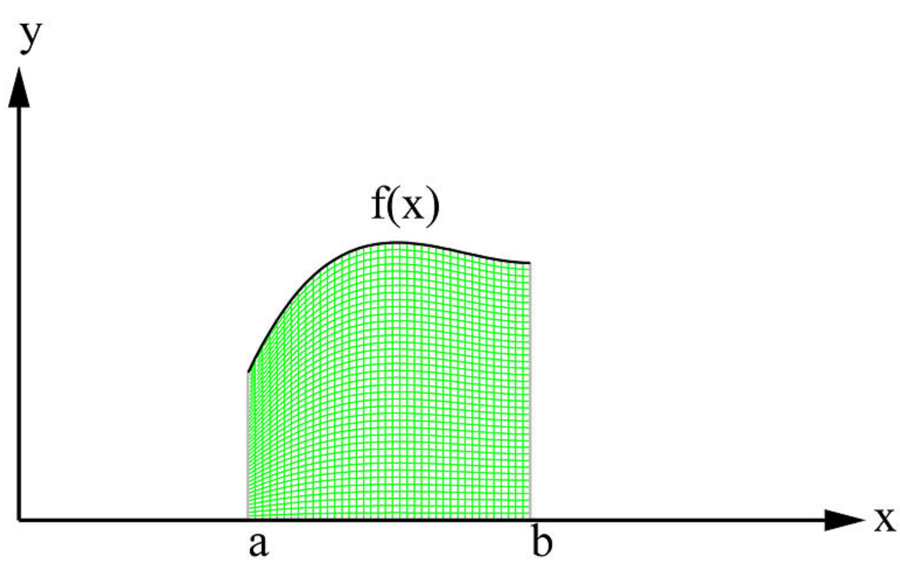


Figure 1a

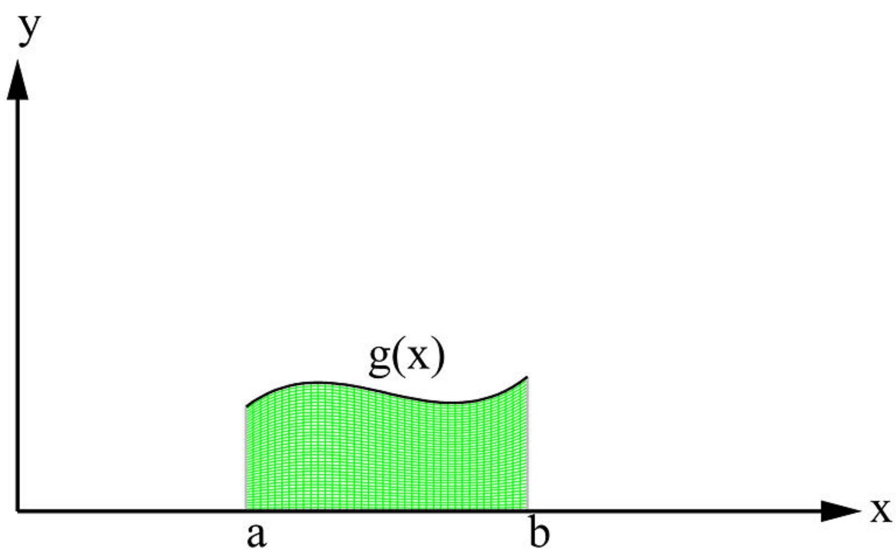


Figure 1b

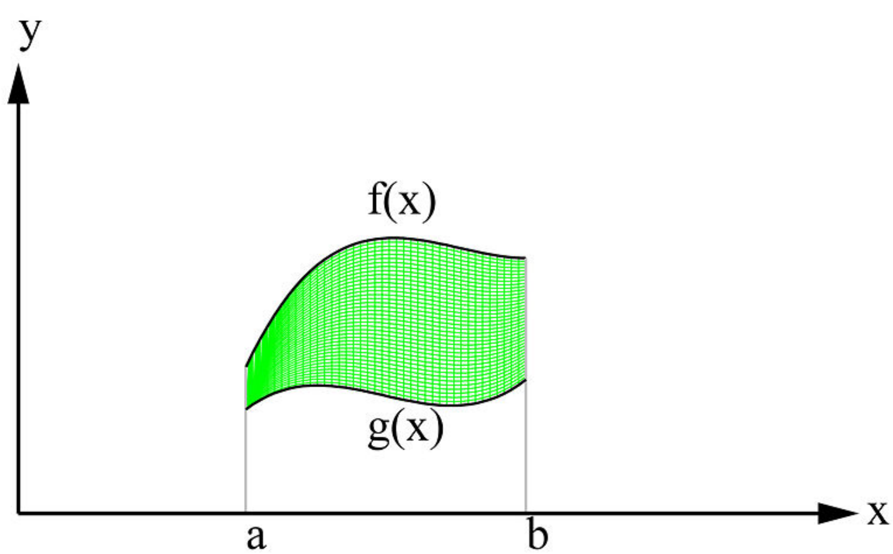


Figure 1c

Therefore, as the graphs show, it makes sense to say that

[Area under f (Fig. 1a)] $-$ [Area under g (Fig. 1b)] $=$ [Area between f and g (Fig. 1c)],

$$\int_a^b f(x)dx - \int_a^b g(x) = \int_a^b [f(x) - g(x)]dx.$$

This relation is valid as long as the two functions are continuous and the upper function $f(x) \geq g(x)$ on the interval $[a,b]$.

The Area Between Two Curves (*With respect to the $x-$ axis*)

If f and g are two continuous functions on the interval $[a,b]$ and $f(x) \geq g(x)$ for all values of x in the interval, then the area of the region that is bounded by the two functions is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

Example 1:

Find the area of the region enclosed between $y = x^2$ and $y = x + 6$.

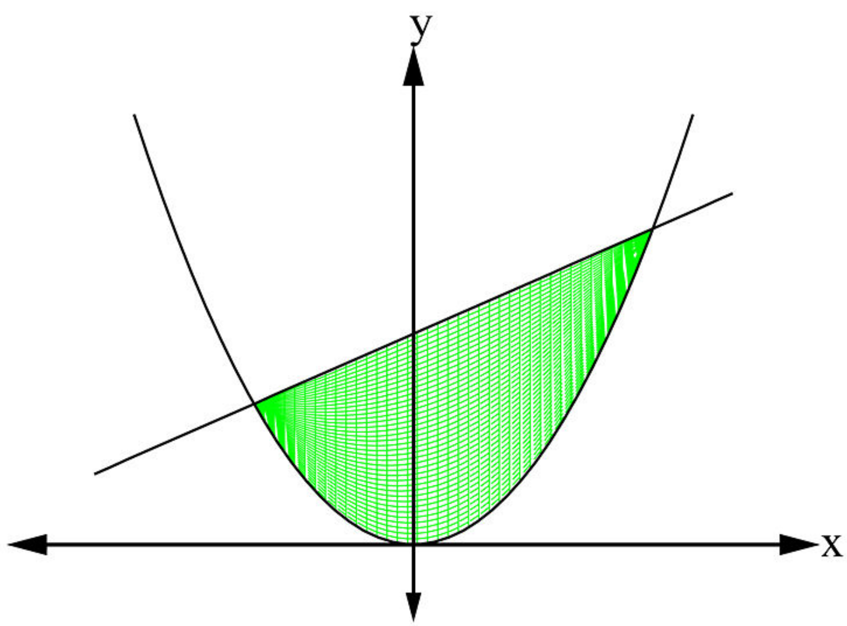


Figure 2

Solution:

We first make a sketch of the region (Figure 2) and find the end points of the region. To do so, we simply equate the two functions,

$$x^2 = x + 6$$

,

and then solve for x.

$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$

from which we get $x = -2$ and $x = 3$.

So the upper and lower boundaries intersect at points $(-2,4)$ and $(3,9)$.

As you can see from the graph, $x + 6 \geq x^2$ and hence $f(x) = x + 6$ and $g(x) = x^2$ in the interval $[-2,3]$. Applying the area formula,

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^3 [(x+6) - (x^2)]dx. \end{aligned}$$

Integrating,

$$\begin{aligned} A &= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^3 \\ &= \frac{125}{6}. \end{aligned}$$

So the area between the two curves $f(x) = x + 6$ and $g(x) = x^2$ is $125/6$.

Sometimes it is possible to apply the area formula with respect to the $y-$coordinates instead of the $x-$coordinates. In this case, the equations of the boundaries will be written in such a way that y is expressed explicitly as a function of x (Figure 3).

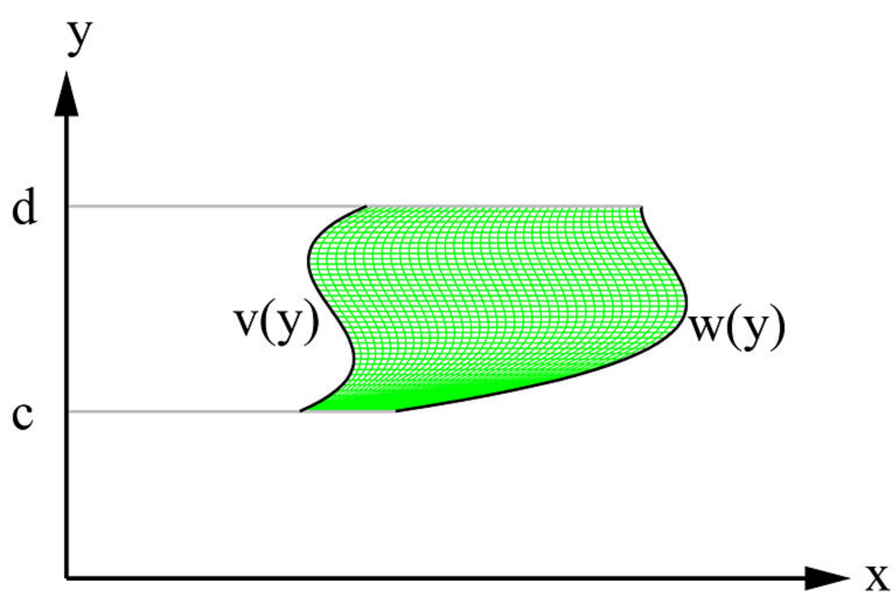


Figure 3

The Area Between Two Curves (*With respect to the y−axis*)

If w and v are two continuous functions on the interval $[c,d]$ and $w(y) \geq v(y)$ for all values of y in the interval, then the area of the region that is bounded by $x = v(y)$ on the left, $x = w(y)$ on the right, below by $y = c$, and above by $y = d$, is given by

$$A = \int_c^d [w(y) - v(y)]dy.$$

Example 2:

Find the area of the region enclosed by $x = y^2$ and $y = x - 6$.

Solution:

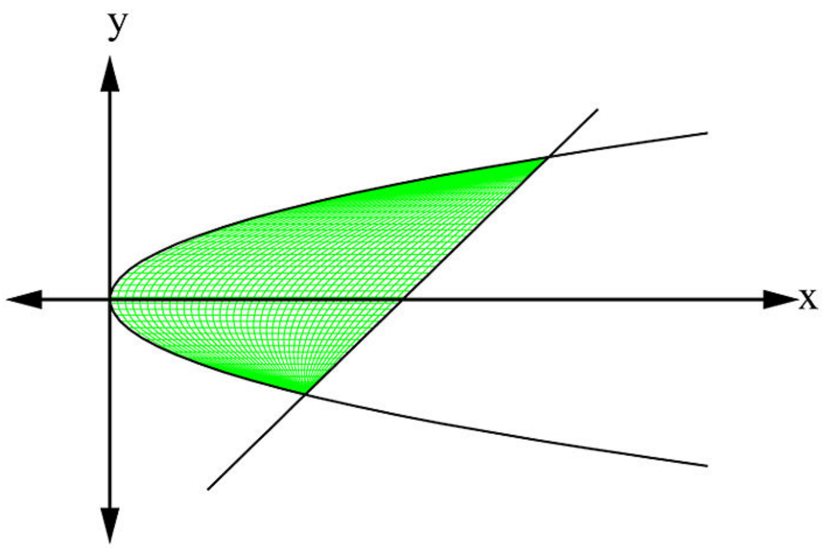


Figure 4

As you can see from Figure 4, the left boundary is $x = y^2$ and the right boundary is $y = x - 6$. The region extends over the interval $-2 \le y \le 3$. However, we must express the equations in terms of y. We rewrite

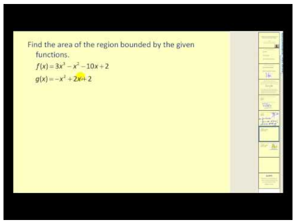
$$\begin{aligned} x &= y^2 \\ x &= y + 6 \end{aligned}$$

Thus

$$\begin{aligned} A &= \int_{-2}^{3} [y + 6 - y^2] dy \\ &= \left[\frac{y^2}{2} + 6y - \frac{y^3}{3} \right]_{-2}^{3} \\ &= \frac{125}{6}. \end{aligned}$$

Multimedia Links

For a video presentation of the area between two graphs **(14.0)(16.0)**, see Math Video Tutorials by James Sousa, Are a Between Two Graphs (6:12).



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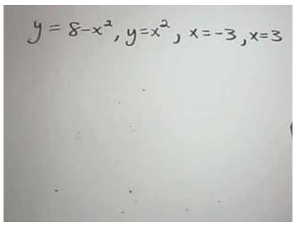
For an additional video presentation of the area between two curves **(14.0)(16.0)**, see Just Math Tutoring, Finding Areas Between Curves (9:51).



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Review Questions

In problems #1 - 7, sketch the region enclosed by the curves and find the area.

1. $y = x^2, y = \sqrt{x}$, on the interval $[0.25, 1]$
2. $y = 0, y = \cos 2x$, on the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
3. $y = |-1+x|+2, y = \frac{-1}{5}x+7$
4. $y = \cos x, y = \sin x, x = 0, x = 2\pi$
5. $x = y^2, y = x-2$, integrate with respect to y
6. $y^2 - 4x = 4, 4x - y = 16$ integrate with respect to y
7. $y = 8\cos x, y = sec^2 x, -\pi/3 \leq x \leq \pi/3$
8. Find the area enclosed by $x = y^3$ and $x = y$.
9. If the area enclosed by the two functions $y = k\cos x$ and $y = kx^2$ is 2, what is the value of k?
10. Find the horizontal line $y = k$ that divides the region between $y = x^2$ and $y = 9$ into two equal areas.

5.2 Volumes

Learning Objectives

- Learn the basic concepts of volume and how to compute it with a given cross-section
- Learn how to compute volume by the *disk method*
- Learn how to compute volume by the *washer method*
- Learn how to compute volume by *cylindrical shells*

In this section, we will use definite integrals to find volumes of different solids.

The Volume Formula

A circular cylinder can be generated by translating a circular disk along a line that is perpendicular to the disk (Figure 5). In other words, the cylinder can be generated by moving the cross-sectional area A (the disk) through a distance h. The resulting volume is called the ***volume of solid*** and it is defined to be

$$V = Ah.$$

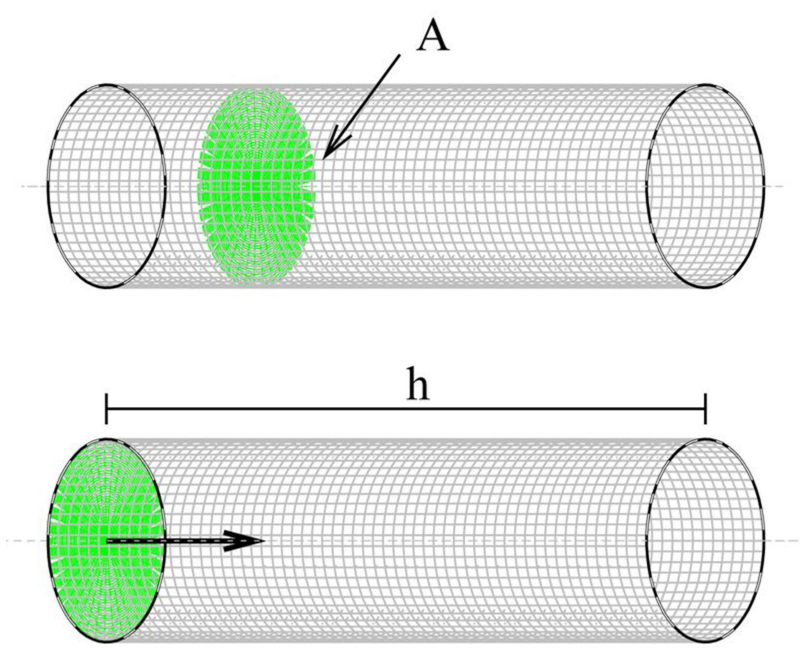


Figure 5

The volume of solid does not necessarily have to be circular. It can take any arbitrary shape. One useful way to find the volume is by a technique called "slicing." To explain the idea, suppose a solid S is positioned on the $x-$axis and extends from points $x = a$ to $x = b$ (Figure 6).

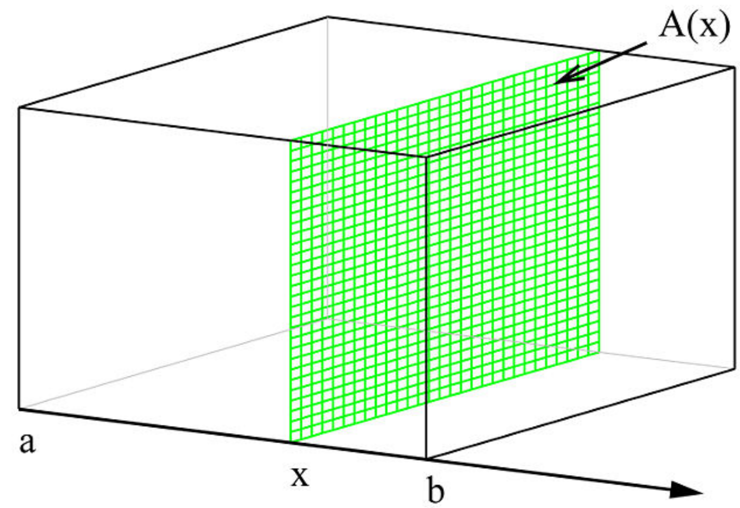


Figure 6

Let $A(x)$ be the cross-sectional area of the solid at some arbitrary point x. Just like we did in calculating the definite integral in the previous chapter, divide the interval $[a,b]$ into n sub-intervals and with widths

$$\triangle x_1, \triangle x_2, \triangle x_3, \ldots, \triangle x_n.$$

Eventually, we get planes that cut the solid into n slices

$$S_1, S_2, S_3, \ldots, S_n.$$

Take one slice, S_k. We can approximate slice S_k to be a rectangular solid with thickness $\triangle x_k$ and cross-sectional area $A(x_k)$. Thus the volume V_k of the slice is approximately

$$V_k \approx A(x_k) \triangle x_k.$$

Therefore the volume V of the entire solid is approximately

$$\begin{aligned} V &= V_1 + V_2 + \ldots + V_n \\ &\approx \sum_{k=1}^{n} A(x_k) \triangle x_k. \end{aligned}$$

If we use the same argument to derive a formula to calculate the area under the curve, let us increase the number of slices in such a way that $\triangle x_k \to 0$. In this case, the slices become thinner and thinner and, as a result, our approximation will get better and better. That is,

$$V = \lim_{\triangle x \to 0} = \sum_{k=1}^{n} A(x_k)\triangle x_k.$$

Notice that the right-hand side is just the definition of the definite integral. Thus

$$\begin{aligned} V &= \lim_{\triangle x \to 0} = \sum_{k=1}^{n} A(x_k)\triangle x_k \\ &= \int_a^b A(x)dx. \end{aligned}$$

The Volume Formula (*Cross-section perpendicular to the* $x-$axis *)*

Let S be a solid bounded by two parallel planes perpendicular to the $x-$axis at $x = a$ and $x = b$. If each of the cross-sectional areas in $[a,b]$ are perpendicular to the $x-$ axis, then the volume of the solid is given by

$$V = \int_a^b A(x)dx.$$

where $A(x)$ is the area of a cross section at the value of x on the $x-$axis.

The Volume Formula (*Cross-section perpendicular to the* $y-$ *axis*)

Let S be a solid bounded by two parallel planes perpendicular to the $y-$axis at $y = c$ and $y = d$. If each of the cross-sectional areas in $[c,d]$ are perpendicular to the $y-$axis, then the volume of the solid is given by

$$V = \int_c^d A(y)dy.$$

where $A(y)$ is the area of a cross section at the value of y on the $y-$axis.

Example 1:

Derive a formula for the volume of a pyramid whose base is a square of sides a and whose height (altitude) is h.

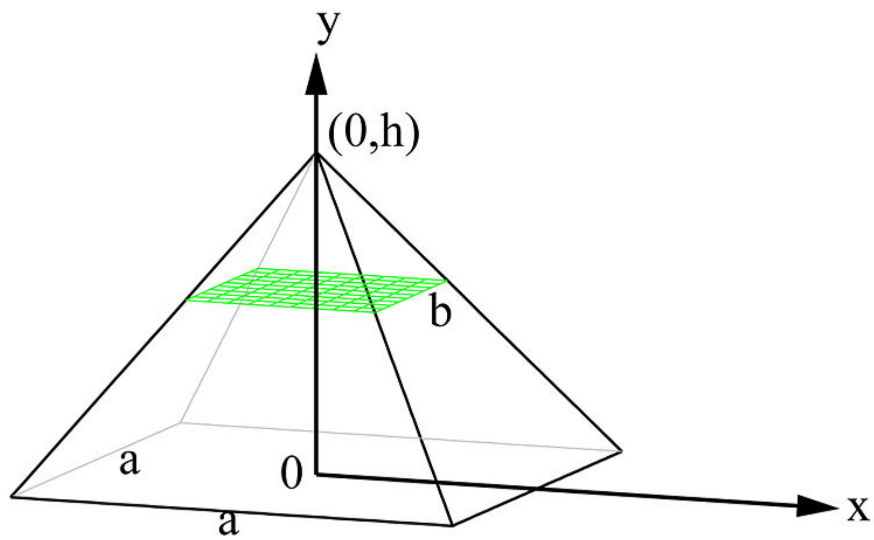


Figure 7a

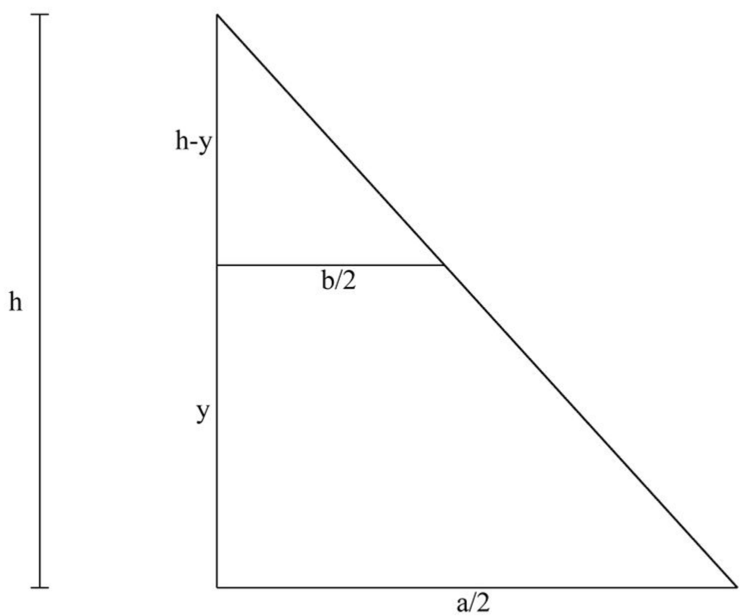


Figure 7b

Solution:

Let the $y-$axis pass through the apex of the pyramid, as shown in Figure (7a). At any point y in the interval $[0,h]$, the cross-sectional area is a square. If b is the length of the sides of any arbitrary square, then, by similar triangles (Figure 7b),

$$\frac{\frac{1}{2}b}{\frac{1}{2}a} = \frac{h-y}{h},$$
$$b = \frac{a}{h}(h-y).$$

Since the cross-sectional area at y is $A(y) = b^2$,

$$A(y) = b^2 = \frac{a^2}{h^2}(h-y)^2.$$

Using the volume formula,

$$\begin{aligned} V &= \int_c^d A(y)dy \\ &= \int_0^h \frac{a^2}{h^2}(h-y)^2 dy \\ &= \frac{a^2}{h^2}\int_0^h (h-y)^2 dy. \end{aligned}$$

Using $u-$substitution to integrate, we eventually get

$$\begin{aligned} V &= \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h \\ &= \frac{1}{3}a^2 h. \end{aligned}$$

Therefore the volume of the pyramid is $V = \frac{1}{3}a^2h$, which agrees with the standard formula.

Volumes of Solids of Revolution

The Method of Disks

Suppose a function f is continuous and non-negative on the interval $[a,b]$, and suppose that R is the region between the curve f and the $x-$axis (Figure 8a). If this region is revolved about the $x-$axis, it will generate a solid that will have circular cross-sections (Figure 8b) with radii of $f(x)$ at each x.

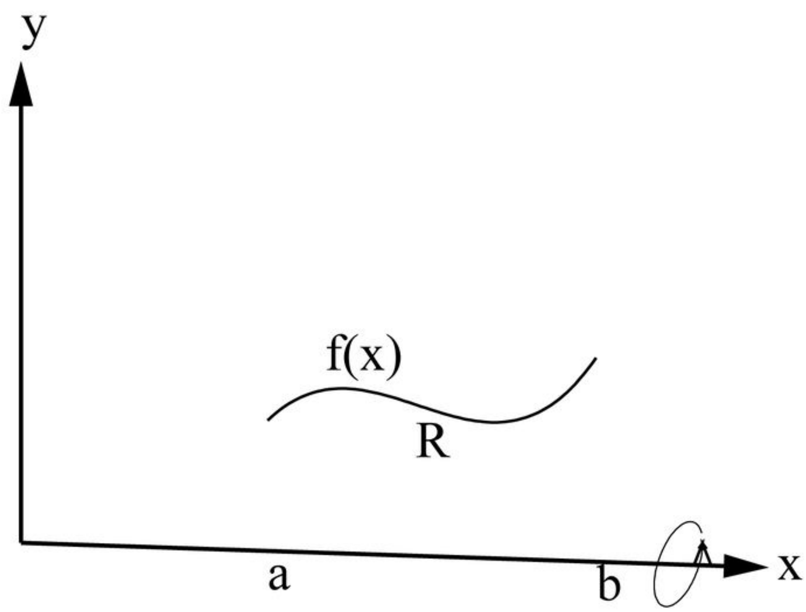


Figure 8a

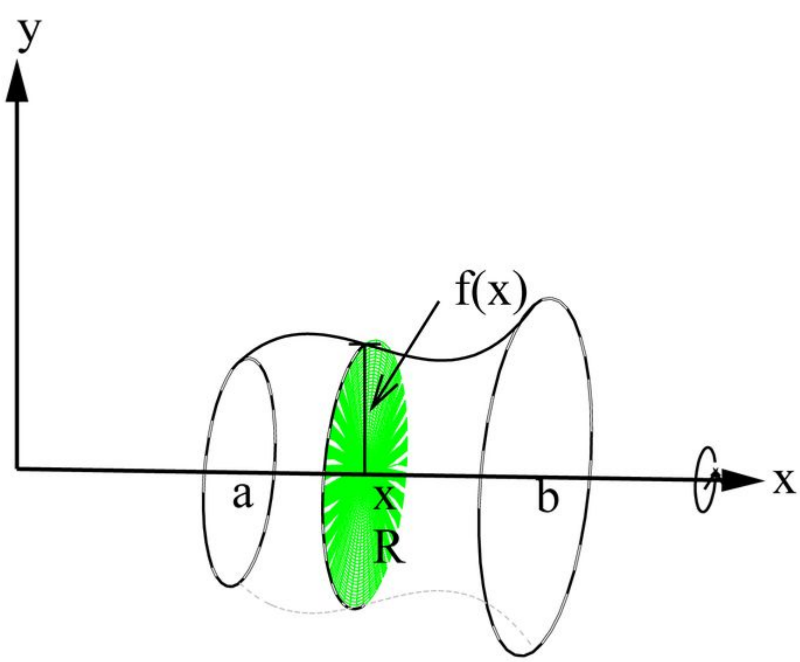


Figure 8b

Each cross-sectional area can be calculated by

$$A(x) = \pi[f(x)]^2.$$

Since the volume is defined as

$$V = \int_a^b A(x)dx,$$

the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2 dx.$$

Volumes by the Method of Disks (*revolution about the $x-$ axis*)

$$V = \int_a^b \pi[f(x)]^2 dx.$$

Because the shapes of the cross-sections are circular or look like the shapes of disks, the application of this method is commonly known as the ***method of disks***.

Example 2

Calculate the volume of the solid that is obtained when the region under the curve $\sqrt{x}$ is revolved about the $x-$axis over the interval $[1, 7]$.

Solution:

As Figures 9a and 9b show, the volume is

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2 dx \\ &= \int_1^7 \pi[\sqrt{x}]^2 dx \\ &= \pi\left[\frac{x^2}{2}\right]_1^7 \\ &= 24\pi. \end{aligned}$$

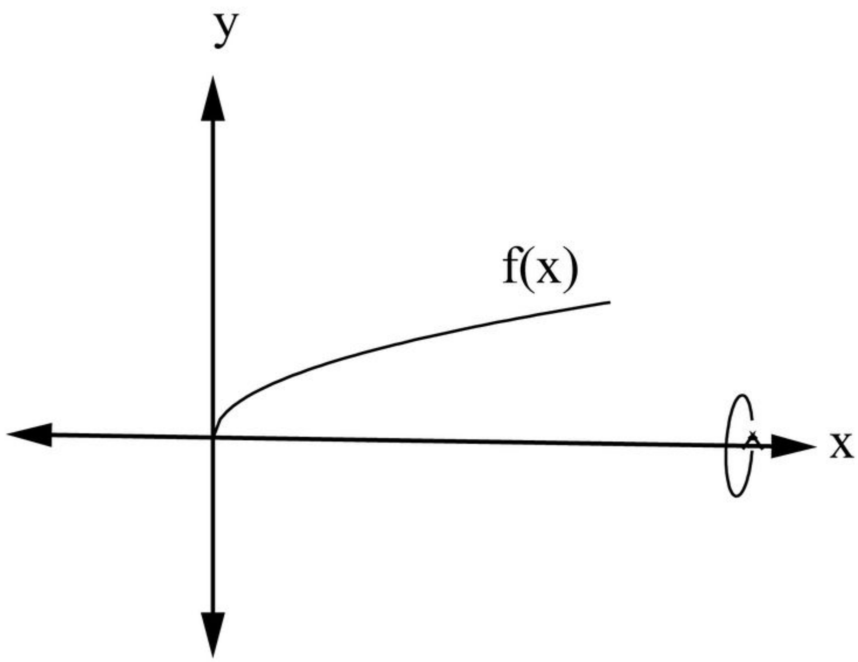


Figure 9a

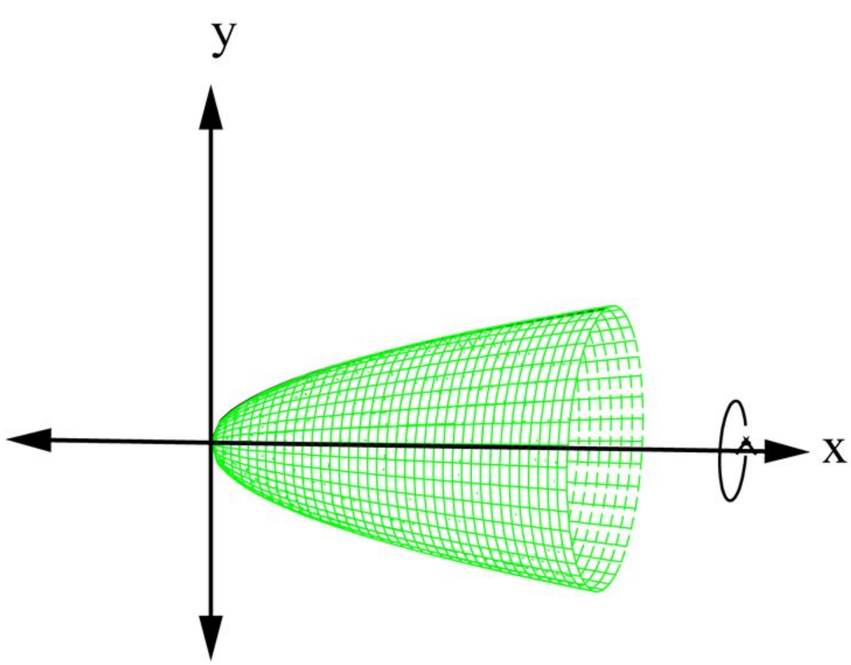


Figure 9b

Example 3:

Derive a formula for the volume of the sphere with radius *r*.

Solution:

One way to find the formula is to use the disk method. From your algebra, a circle of radius r and center at the origin is given by the formula

$$x^2+y^2=r^2$$

If we revolve the circle about the $x-$axis, we will get a sphere. Using the disk method, we will obtain a formula for the volume. From the equation of the circle above, we solve for y:

$$f(x)=y=\sqrt{r^2-x^2},$$

thus

$$\begin{aligned}
V &= \int_a^b \pi[f(x)]^2 dx \\
&= \int_{-r}^{+r} \pi\left[\sqrt{r^2-x^2}\right]^2 dx \\
&= \pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r} \\
&= \frac{4}{3}\pi r^3.
\end{aligned}$$

This is the standard formula for the volume of the sphere.

The Method of Washers

To generalize our results, if f and g are non-negative and continuous functions and

$$f(x)\geq g(x)$$

for

$$a\leq x\leq b,$$

Then let R be the region enclosed by the two graphs and bounded by $x=a$ and $x=b$. When this region is revolved about the $x-$axis, it will generate washer-like cross-sections (Figures 10a and 10b). In this case, we will have two radii: an inner radius $g(x)$ and an outer radius $f(x)$. The volume can be given by:

$$V(x)=\int_a^b \pi\left([f(x)]^2-[g(x)]^2\right)dx.$$

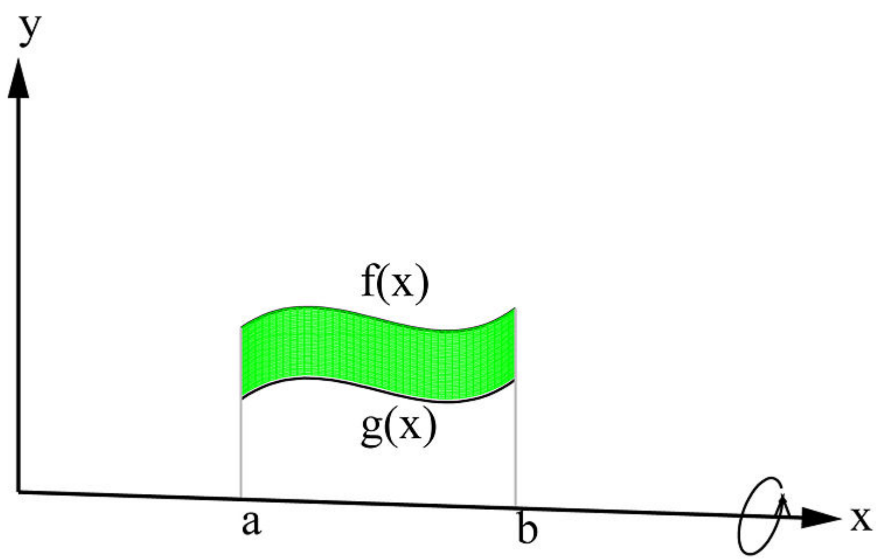


Figure 10a

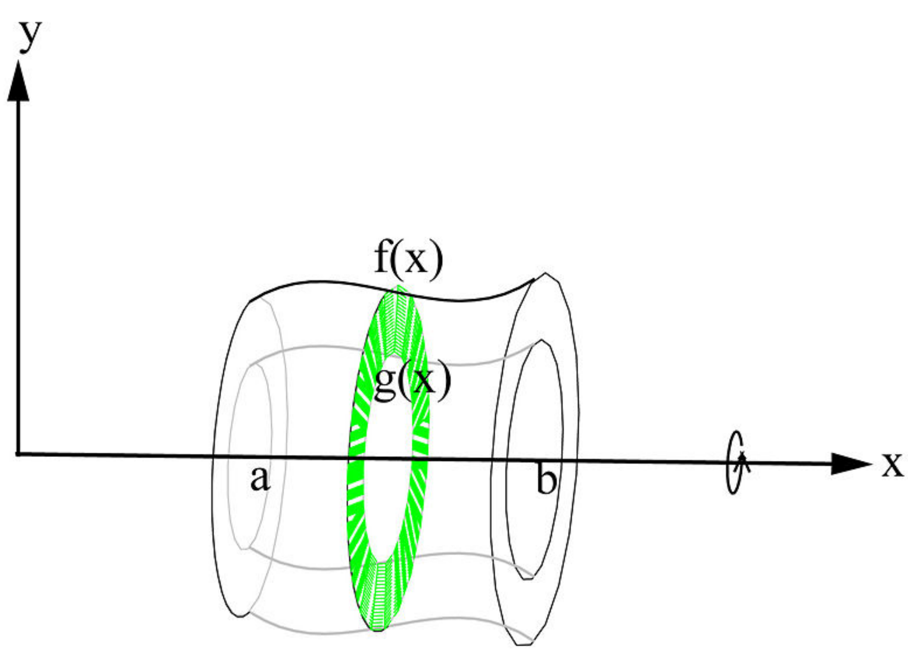


Figure 10b

Volumes by the Method of Washers (*revolution about the x− axis*)

$$V(x) = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx.$$

Example 4:

Find the volume generated when the region between the graphs $f(x) = x^2 + 1$ and $g(x) = x$ over the interval $[0,3]$ is revolved about the $x-$axis.

Solution:

As Figures 11a and 11b show, the volume is

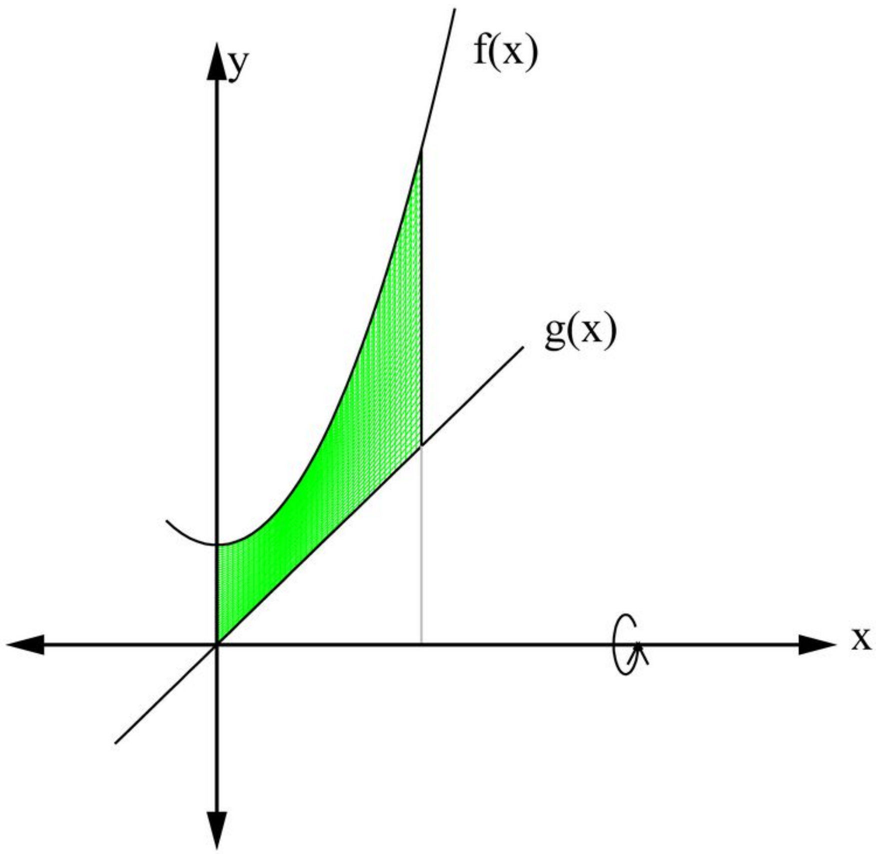


Figure 11a

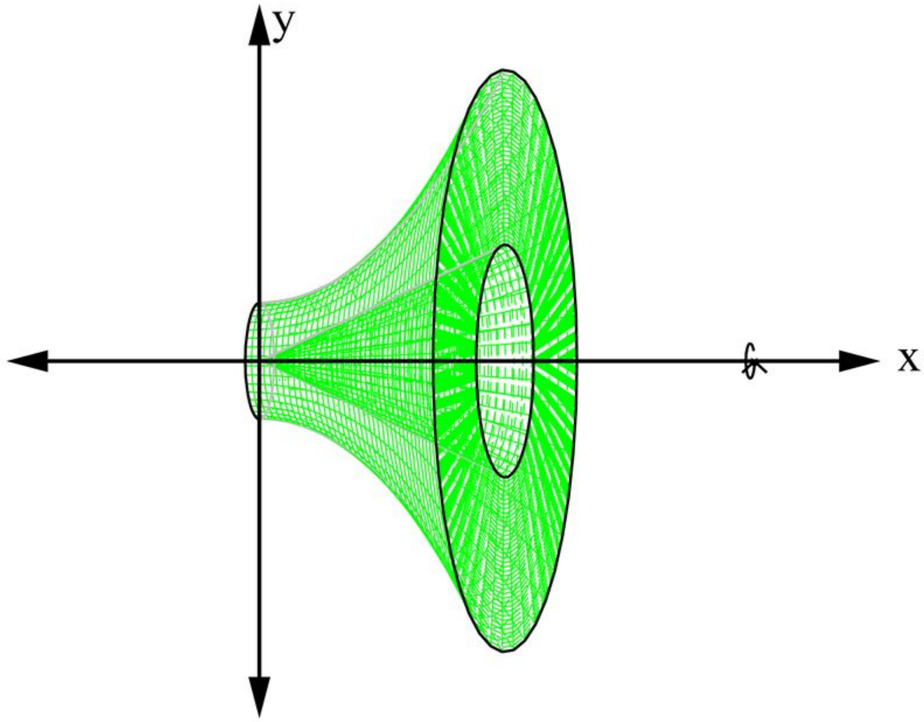


Figure 11b

From the formula above,

$$\begin{aligned} V(x) &= \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx \\ &= \int_0^3 \pi \left((x^2+1)^2 - (x)^2 \right) dx \\ &= \int_0^3 \pi \left(x^4 + x^2 + 1 \right) dx \\ &= \frac{303\pi}{5}. \end{aligned}$$

The methods of disks and washers can also be used if the region is revolved about the $y-$axis. The analogous formulas can be easily deduced from the above formulas or from the volumes of solids generated.

Disks:

$$V = \int_c^d \pi [u(y)]^2 dy.$$

Washers:

$$V = \int_c^d \pi \left([w(y)]^2 - [v(y)]^2 \right) dy.$$

Example 5:

What is the volume of the solid generated when the region enclosed by $y = \sqrt{x}$,$y = 3$, and $x = 0$ is revolved about the $y-$axis?

Solution:

Since the solid generated is revolved about the $y-$axis (Figures 12a and 12b), we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus $u(y) = y^2$. The volume is

$$\begin{aligned}
V &= \int_c^d \pi[u(y)]^2 dy \\
&= \int_0^3 \pi[y^2]^2 dy \\
&= \int_0^3 \pi y^4 dx \\
&= \pi \left[\frac{y^5}{5}\right]_0^3 \\
&= \pi \left[\frac{3^5}{5} - 0\right] \\
&= \frac{243\pi}{5}.
\end{aligned}$$

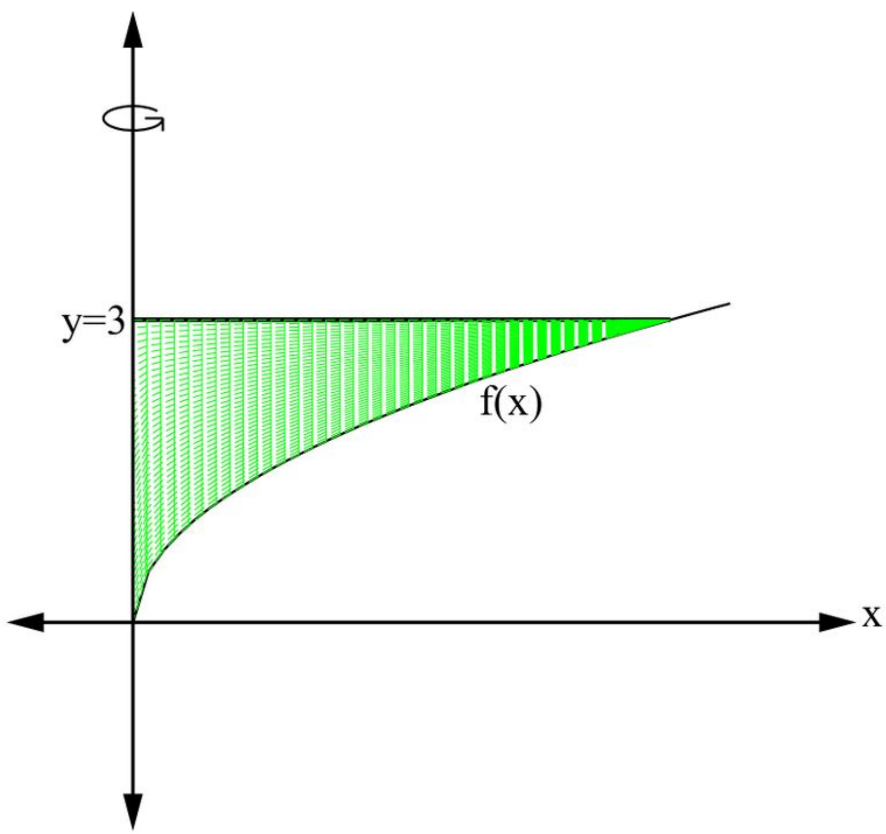


Figure 12a

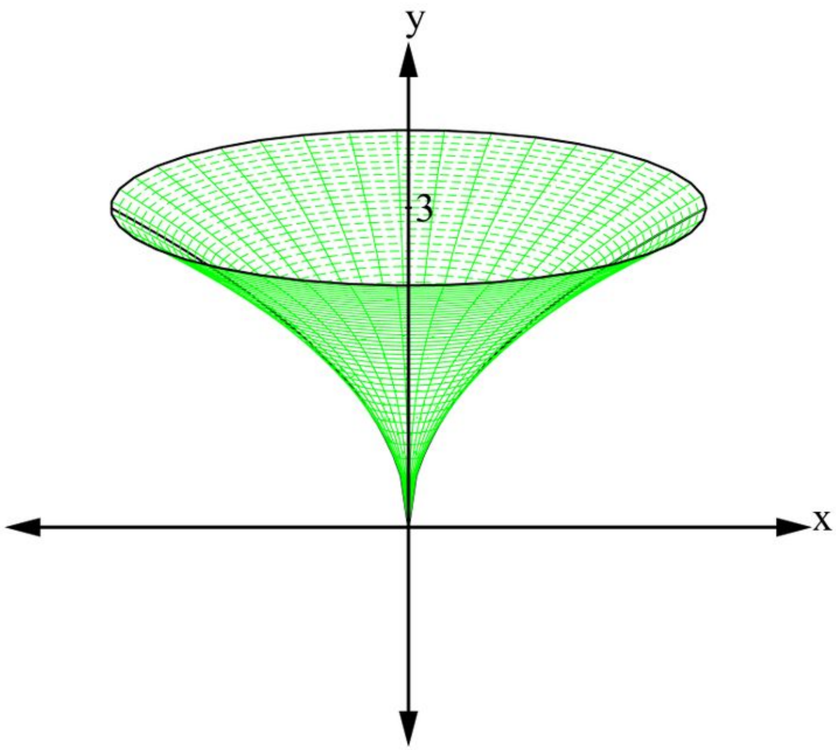


Figure 12b

Volume By Cylindrical Shells

The method of computing volumes so far depended upon computing the cross-sectional area of the solid and then integrating it across the solid. What happens when the cross-sectional area cannot be found or the integration is too difficult to solve? Here is where the ***shell method*** comes along.

To show how difficult it sometimes is to use the disk or the washer methods to compute volumes, consider the region enclosed by the function $f(x) = x - x^2$. Let us revolve it about the line $x = -1$ (Figure 13a) to generate the shape of a doughnut-shaped cake (Figure 13b). What is the volume of this solid?

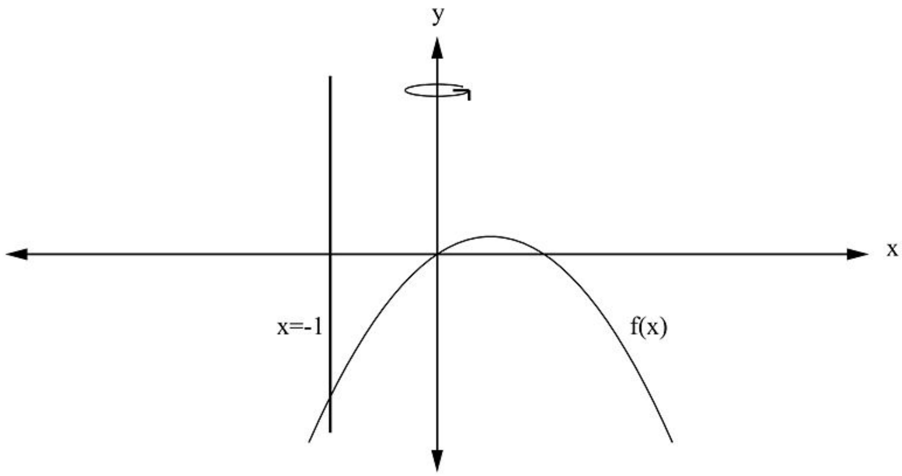


Figure 13a

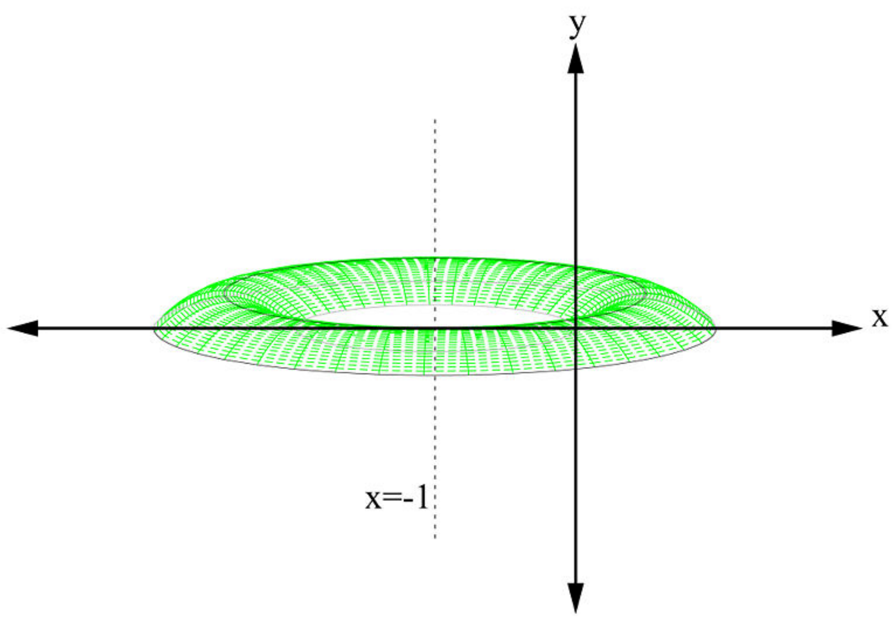


Figure 13b

If we wish to integrate with respect to the $y-$axis, we have to solve for x in terms of y. That would not be easy (try it!). An easier way is to integrate with respect to the $x-$axis by using the shell method. Here is how: A cylindrical shell is a solid enclosed by two concentric cylinders. If the inner radius is r_1 and the outer one is r_2, with both of height h, then the volume is (Figure 14)

$$\begin{aligned} V &= [\text{area of the cross-section}] \cdot [\text{height}] \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \cdot \left[\frac{1}{2}(r_2 + r_1)\right] \cdot h \cdot (r_2 - r_1). \end{aligned}$$

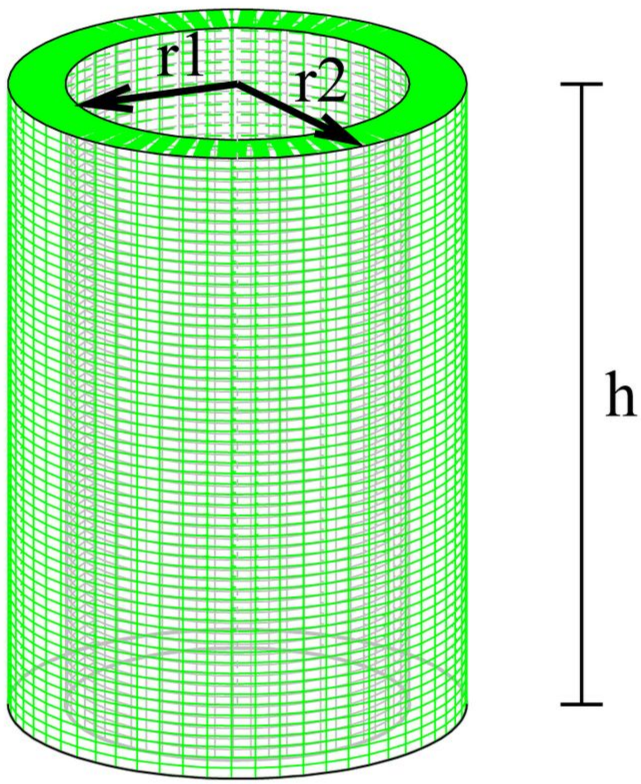


Figure 14

Notice however that $(r_2 - r_1)$ is the thickness of the shell and $\frac{1}{2}(r_2 + r_1)$ is the average radius of the shell.

Thus

$$V = 2\pi \cdot [\text{average radius}] \cdot [\text{height}] \cdot [\text{thickness}].$$

Replacing the average radius with a single variable r and using h for the height, we have

$$V = 2\pi \cdot r \cdot h \cdot [\text{thickness}].$$

In general the shell's thickness will be dx or dy depending on the axis of revolution. This discussion leads to the following formulas for rotation about an axis. We will then use this formula to compute the volume V of the solid of revolution that is generated by revolving the region about the $x-$axis.

Volume By Cylindrical Shell about the $y-$Axis

Suppose f is a continuous function in the interval $[a, b]$ and the region R is bounded above by $y = f(x)$ and below by the $x-$axis, and on the sides by the lines $x = a$ and $x = b$. If R is rotated around the $y-$axis, then the cylinders are vertical, with $r = x$ and $h = f(x)$. The volume of the solid is given by

$$V = \int_a^b 2\pi rhdx = \int_b^a 2\pi x f(x)dx.$$

Volume By Cylindrical Shell about the $x-$ Axis

Equivalently, if the volume is generated by revolving the same region about the $x-$axis, then the cylinders are horizontal with

$$v = \int_c^d 2\pi rh dy,$$

where $c = f^{-1}(a)$ and $d = f^{-1}(b)$. The values of r and h are determined in context, as you will see in Example 6.

Note: Example 7 shows what to do when the rotation is not about an axis.

Example 6:

A solid figure is created by rotating the region R (Figure 15) around the $x-$axis. R is bounded by the curve $y = x^2$ and the lines $x = 0$ and $x = 2$. Use the shell method to compute the volume of the solid.

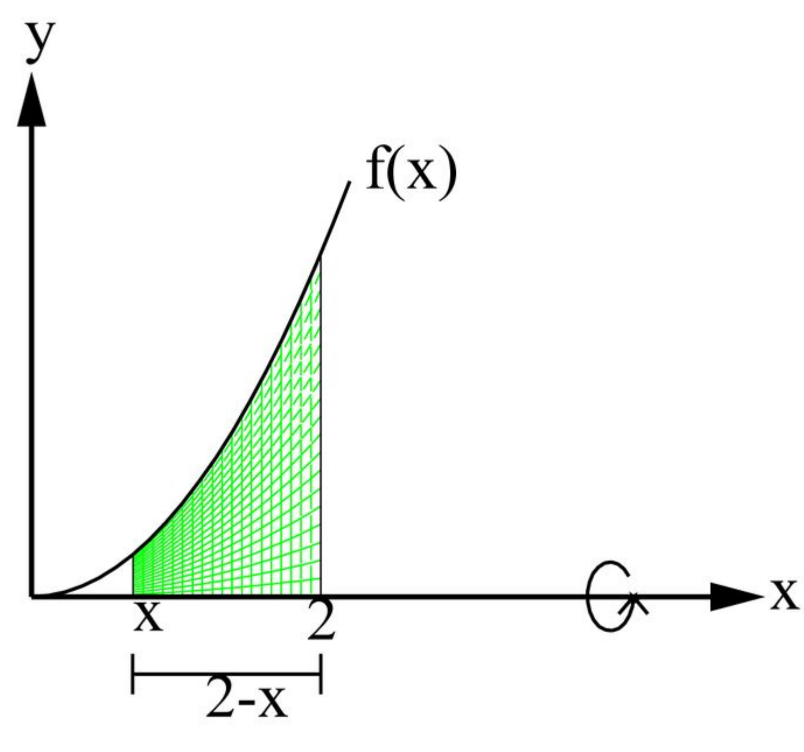


Figure 15

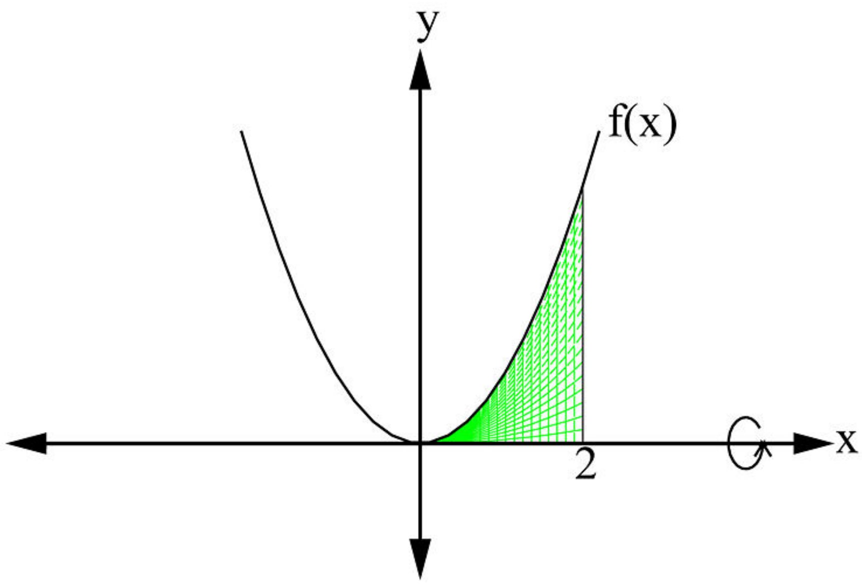


Figure 16

Solution:

From Figure 15 we can identify the limits of integration: y runs from 0 to 4. A horizontal strip of this region would generate a cylinder with height $2-\sqrt{y}$ and radius y. Thus the volume of the solid will be

$$\begin{aligned} V &= \int_c^d 2\pi rh dy \\ &= \int_0^4 2\pi y(2-\sqrt{y})dy \\ &= 2\pi \int_0^4 (2y - y^{3/2})dy \\ &= 2\pi \left[y^2 - \frac{2}{5}y^{5/2} \right]_0^4 \\ &= \frac{32\pi}{5}. \end{aligned}$$

Note: The alert reader will have noticed that this example could be worked with a simpler integral using disks. However, the following example can only be solved with shells.

Example 7:

Find the volume of the solid generated by revolving the region bounded by $y = x^3 + \frac{1}{2}x + \frac{1}{4}, y = \frac{1}{4}$, and $x = 1$, about $x = 3$.

Solution:

As you can see, the equation $y = x^3 + \frac{1}{2}x + \frac{1}{4}$ cannot be easily solved for x and therefore it will be necessary to solve the problem by the shell method. We are revolving the region about a line parallel to the $y-$axis and thus integrate with respect to x. Our formula is

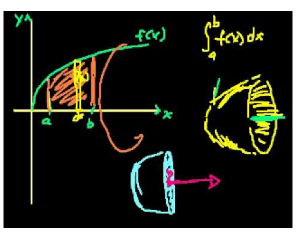
$$V = \int_a^b 2\pi rh dx.$$

In this case, the radius is $3-x$ and the height is $x^3+\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}$. Substituting,

$$\begin{aligned}
V &= 2\pi \int_0^1 (3-x)\left(x^3+\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}\right)dx \\
&= 2\pi \int_0^1 \left(-x^4+3x^3-\frac{1}{2}x^2+\frac{3}{2}x\right)dx \\
&= 2\pi \left[\frac{-1}{5}x^5+\frac{3}{4}x^4-\frac{1}{6}x^3+\frac{3}{4}x^2\right]_0^1 \\
&= 2\pi \left[\frac{-1}{5}+\frac{3}{4}-\frac{1}{6}+\frac{3}{4}\right] \\
&= 2\pi \left[\frac{17}{15}\right] \\
&= \frac{34\pi}{15}.
\end{aligned}$$

Multimedia Links

The following applet allows you to try out solids of revolution about the x-axis for any two functions. You can try inputting the examples above to test it out, and then experiment with new functions and changing the bounds. Volumes of Revolution Applet . In the following video the narrator walks trough the steps of setting up a volume integration **(14.0)(16.0)**. Khan Academy Solids of Revolution (10:05).



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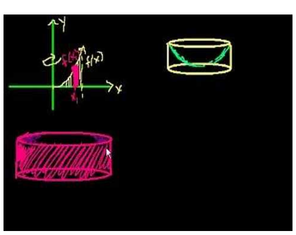
Sometimes the same volume problem can be solved in two different ways **(14.0)(16.0)**. In these two videos, the narrator first finds a volume using shells Khan Academy Solid of Revolution (Part 5) (9:29)



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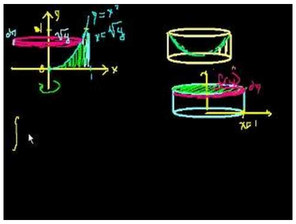
, and then he does the same volume problem using disks. Khan Academy Solid of Revolution (Part 6) (9:19).



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Together these videos show how both methods can be used to solve the same problem (though it's not always done this easily!).

Review Questions

In problems #1 - 4, find the volume of the solid generated by revolving the region bounded by the curves about the $x-$axis.

1. $y = \sqrt{9-x^2}, y = 0$
2. $y = 3+x, y = 1+x^2$
3. $y = \sec x, y = \sqrt{2}, -\pi/4 \le x \le \pi/4$
4. $y = 1, y = x, x = 0$

In problems #5–8, find the volume of the solid generated by revolving the region bounded by the curves about the $y-$axis.

5. $y = x^3, x = 0, y = 1$
6. $x = y^2, y = x-2$
7. $x = \csc y, y = \pi/4, y = 3\pi/4, x = 0$
8. $y = 0, y = \sqrt{x}, x = 4$

In problems #9–12, use cylindrical shells to find the volume generated when the region bounded by the curves is revolved about the axis indicated.

9. $y = \frac{1}{x}, y = 0, x = 1, x = 3$, about the $y-$axis
10. $y = x^2, x = 1, y = 0$, about the $x-$axis
11. $y = 2x-1, y = -2x+3, x = 2$, about the $y-$axis
12. $y^2 = x, y = 1, x = 0$, about the $x-$axis.
13. Use the cylindrical shells method to find the volume generated when the region is bounded by $y = x^3, y = 1, x = 0$ is revolved about the line $y = 1$.

5.3 The Length of a Plane Curve

Learning Objectives

A student will be able to:

- Learn how to find the length of a plane curve for a given function.

In this section will consider the problem of finding the length of a plane curve. Formulas for finding the arcs of circles appeared in early historical records and they were known to many civilizations. However, very little was known about finding the lengths of general curves, such as the length of the curve $y = x^2$ in the interval $[0,2]$, until the discovery of calculus in the seventeenth century.

In calculus, we define an ***arc length*** as the length of a plane curve $y = f(x)$ over an interval $[a,b]$ (Figure 17). When the curve $f(x)$ has a continuous first derivative f' on $[a,b]$, we say that f is a smooth function (or smooth curve) on $[a,b]$.

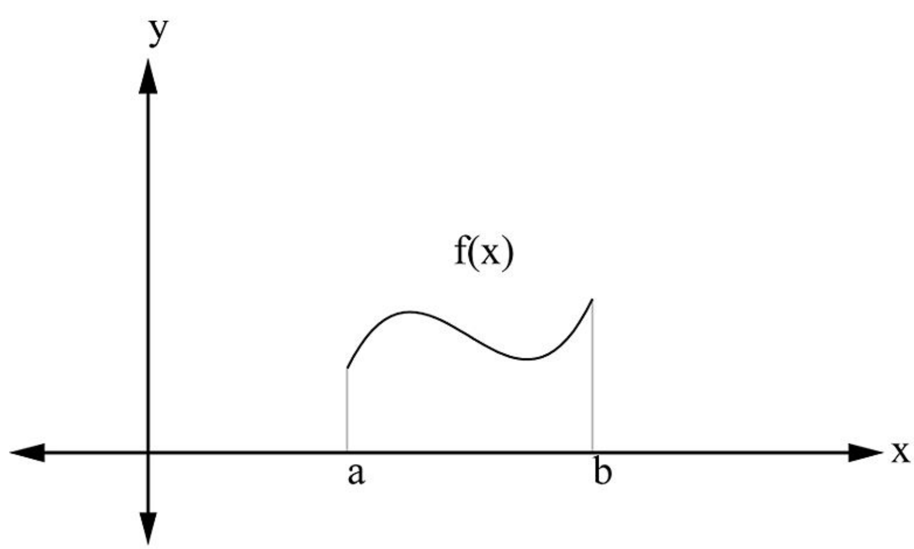


Figure 17

The Arc Length Problem

If $y = f(x)$ is a smooth curve on the interval $[a,b]$, then the arc length L of this curve is defined as

$$L = \int_a^b \sqrt{1+[f'(x)]^2}dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

Example 1:

Find the arc length of the curve $y = x^{\frac{3}{2}}$ on $[1,3]$ (Figure 18).

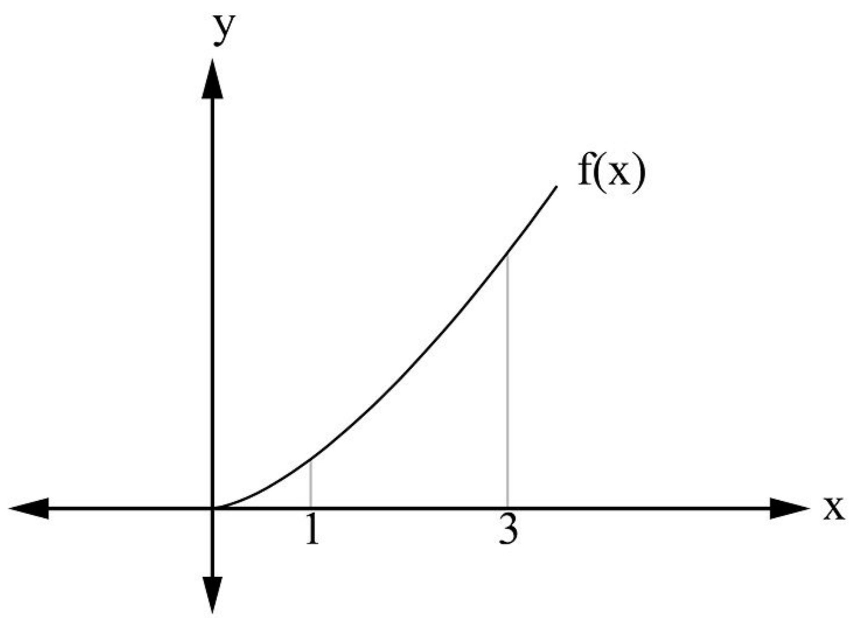


Figure 18

Solution:

Since $y = x^3/2$,

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}.$$

Using the formula above, we get

$$\begin{aligned}\int_a^b \sqrt{1+[f'(x)]^2}dx &= \int_1^3 \sqrt{1+\left[\frac{3}{2}x^{1/2}\right]^2}dx \\ &= \int_1^3 \sqrt{1+\frac{9}{4}x}dx.\end{aligned}$$

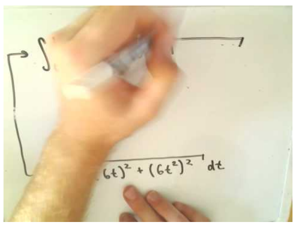
Using $u-$substitution by letting $u = 1+\frac{9}{4}x$, then $du = \frac{9}{4}dx$. Substituting, and remembering to change the limits of integration,

$$\begin{aligned}L &= \frac{4}{9}\int_{13/4}^{31/4} \sqrt{u}du \\ &= \frac{8}{27}\left[u^{3/2}\right]_{13/4}^{31/4} \\ &\approx 4.65.\end{aligned}$$

Multimedia Links

The formula you just used to find the length of a curve was derived by using line segments to approximate the curve. The derivation of that formula can be found at Wikipedia Entry on Arc Length . In the following applet you can explore this further. Experiment with various curves and change the number of segments to see how changing the number of segments is related to approximating the arc length. Arc Length Applet .

For video presentations showing how to obtain the arc length using parametric curves **(16.0)**, see Just Math Tutoring, Arc Length Using Parametric Curves, Example 1 (8:17)



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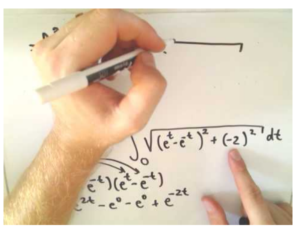
and Just Math Tutoring, Arc Length Using Parametric Curves, Example 2 (7:27).



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Review Questions

1. Find the arc length of the curve

$$y = \frac{(x^2+2)^{3/2}}{3}$$

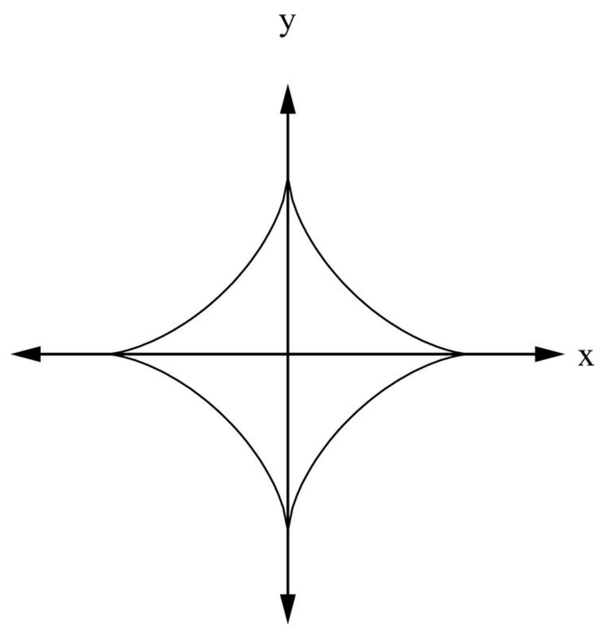
   on $[0,3]$.
2. Find the arc length of the curve

$$x = \frac{1}{6}y^3 + \frac{1}{2y}$$

   on $y \in [1,2]$.
3. Integrate

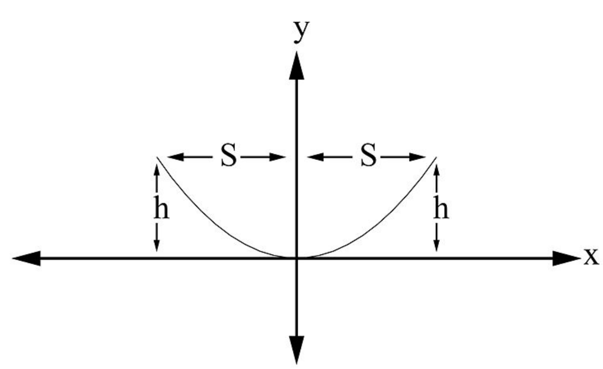
$$x = \int_0^y \sqrt{\sec^4 t - 1}\,dt, \ -\frac{\pi}{4} \le y \le \frac{\pi}{4}.$$

4. Find the length of the curve shown in the figure below. The shape of the graph is called the *astroid* because it looks like a star. The equation of its graph is $x^{2/3} + y^{2/3} = 1$.



5. The figure below shows a suspension bridge. The cable has the shape of a parabola with equation $kx^2 = y$. The suspension bridge has a total length of $2S$ and the height of the cable is h at each end. Show that the total length of the cable is

$$L = 2\int_0^s \sqrt{1 + \frac{4h^2}{S^4}x^2}dx.$$



5.4 Area of a Surface of Revolution

Learning Objectives

A student will be able to:

- Learn how to find the area of a surface that is generated by revolving a curve about an axis or a line.

In this section we will deal with the problem of finding the area of a surface that is generated by revolving a curve about an axis or a line. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter (Figure 19) and the circular cylinder can be generated by revolving a line segment about any axis that is parallel to it (Figure 20).

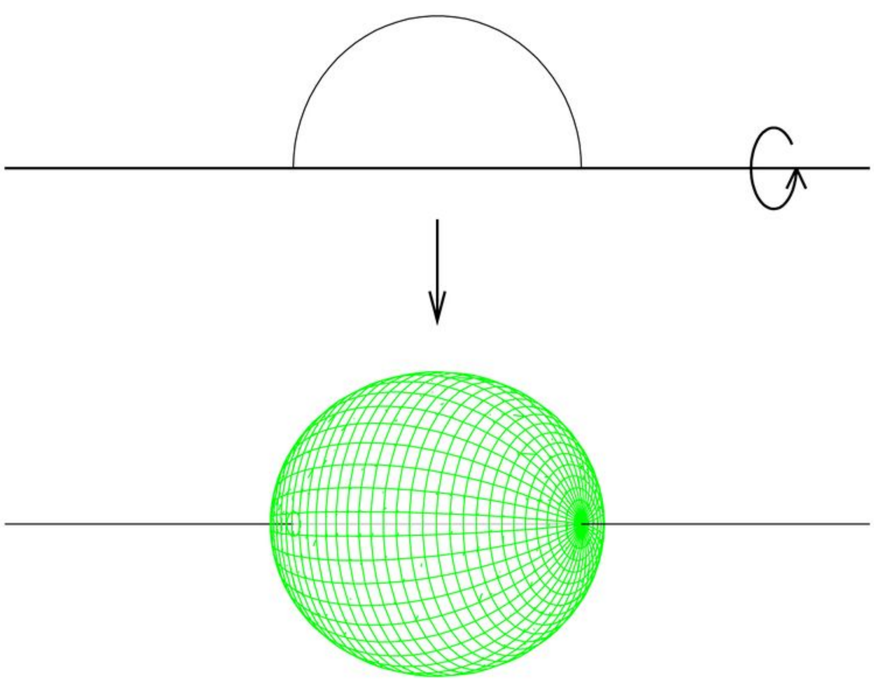


Figure 19

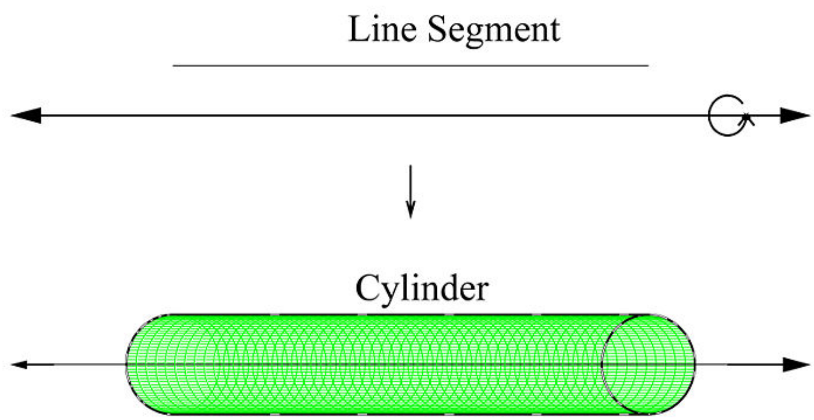


Figure 20

Area of a Surface of Revolution

If f is a smooth and non-negative function in the interval $[a,b]$, then the surface area S generated by revolving the curve $y = f(x)$ between $x = a$ and $x = b$ about the $x-$axis is defined by

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

Equivalently, if the surface is generated by revolving the curve about the $y-$axis between $y = c$ and $y = d$, then

$$S = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}dy = \int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy.$$

Example 1:

Find the surface area that is generated by revolving $y = x^3$ on $[0,2]$ about the $x-$axis (Figure 21).

Solution:

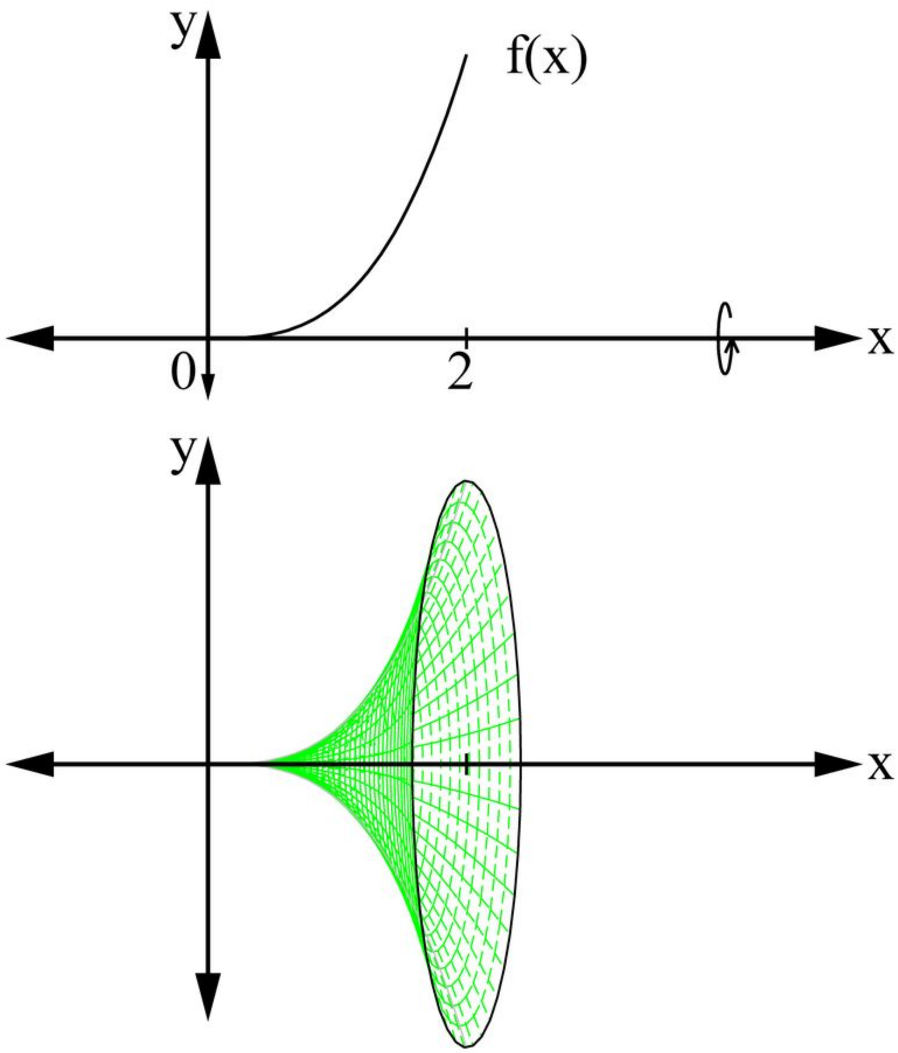


Figure 21

The surface area S is

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\ &= \int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}dx \\ &= 2\pi \int_0^2 x^3(1+9x^4)^{1/2}dx. \end{aligned}$$

Using $u-$substitution by letting $u = 1+9x^4$,

$$\begin{aligned} S &= 2\pi \int_1^{145} u^{1/2} \frac{du}{36} \\ &= \frac{2\pi}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{145} \\ &= \frac{2\pi}{36}\cdot\frac{2}{3}\left[(145)^{3/2}-1\right] \\ &\approx \frac{4\pi}{108}[1745] \\ &\approx 203 \end{aligned}$$

Example 2:

Find the area of the surface generated by revolving the graph of $f(x) = x^2$ on the interval $[0,\ \sqrt{3}]$ about the $y-$axis (Figure 22).

Solution:

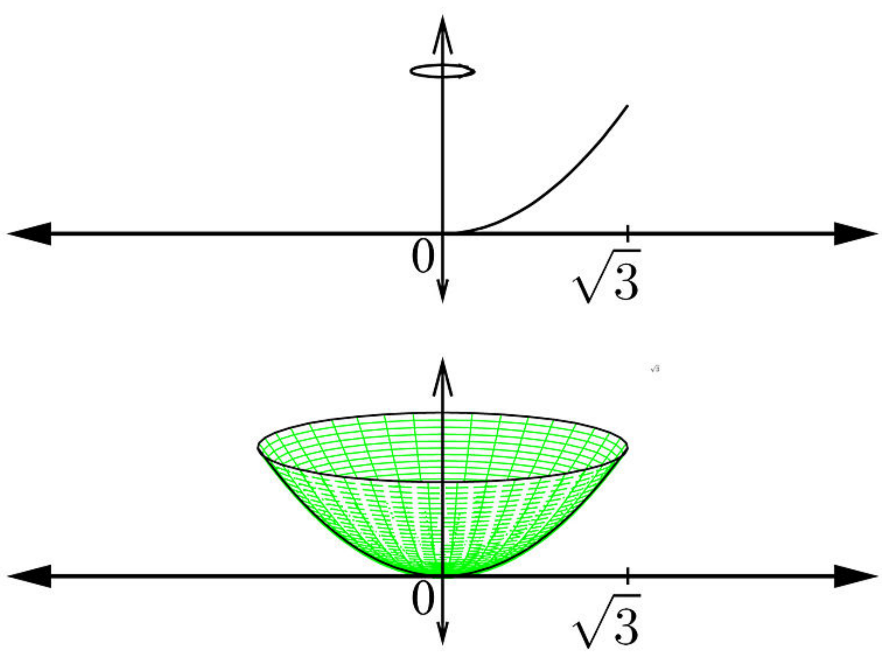


Figure 22

Since the curve is revolved about the $y-$axis, we apply

$$S = \int_c^d 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy.$$

So we write $y = x^2$ as $x = \sqrt{y}$. In addition, the interval on the $x-$axis $[0,\ \sqrt{3}]$ becomes $[0,3]$. Thus

$$S = \int_0^3 2\pi \sqrt{y} \sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy.$$

Simplifying,

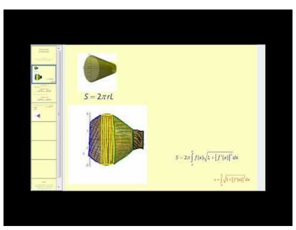
$$S = \pi \int_0^3 \sqrt{4y+1}dy.$$

With the aid of $u-$substitution, let $u = 4y+1$,

$$\begin{aligned} S &= \frac{\pi}{4}\int_1^{13} u^{1/2}du \\ &= \frac{\pi}{6}\left[(13)^{3/2}-1\right] \\ &= \frac{\pi}{6}[46.88-1] \\ &\approx 24 \end{aligned}$$

Multimedia Links

For video presentations of finding the surface area of revolution **(16.0)**, see Math Video Tutorials by James Sousa, Surface Area of Revolution, Part 1 (9:47)



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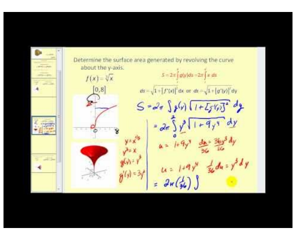
and Math Video Tutorials by James Sousa, Surface Area of Revolution, Part 2 (5:43).



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Review Questions

In problems #1 - 3 find the area of the surface generated by revolving the curve about the $x-$axis.

1. $y = 3x, 0 \leq x \leq 1$
2. $y = \sqrt{x}, 1 \leq x \leq 9$
3. $y = \sqrt{4-x^2}, -1 \leq x \leq 1$

In problems #4–6 find the area of the surface generated by revolving the curve about the $y-$axis.

4. $x = 7y + 2, 0 \leq y \leq 3$
5. $x = y^3, 0 \leq y \leq 8$
6. $x = \sqrt{9 - y^2}, -2 \leq y \leq 2$
7. Show that the surface area of a sphere of radius r is $4\pi r^2$.
8. Show that the lateral area S of a right circular cone of height h and base radius r is

$$S = \pi r \sqrt{r^2 + h^2}.$$

5.5 Applications from Physics, Engineering, and Statistics

Learning Objectives

A student will be able to:

- Learn how to apply definite integrals to several applications from physics, engineering, and applied mathematics such as work, fluids statics, and probability.

In this section we will show how the definite integral can be used in different applications. Some of the concepts may sound new to the reader, but we will explain what you need to comprehend as we go along. We will take three applications: The concepts of work from physics, fluid statics from engineering, and the normal probability from statistics.

Work

Work in physics is defined as the product of the force and displacement. Force and displacement are vector quantities, which means they have a direction and a magnitude. For example, we say the compressor exerts a force of 200 Newtons (N) upward. The magnitude here is 200 N and the direction is upward. Lowering a book from an upper shelf to a lower one by a distance of 0.5 meters away from its initial position is another example of the vector nature of the displacement. Here, the magnitude is 0.5 m and the direction is downward, usually indicated by a minus sign, i.e., a displacement of -0.5 m. The product of those two vector quantities (called the ***inner product***, see Chapter 10) gives the work done by the force. Mathematically, we say

$$W = Fd,$$

where F is the force and d is the displacement. If the force is measured in Newtons and distance is in meters, then work is measured in the units of energy which is in joules (J).

Example 1:

You push an empty grocery cart with a force of 44 N for a distance of 12 meters. How much work is done by you (the force)?

Solution:

Using the formula above,

$$\begin{aligned} W &= Fd \\ &= (44)(12) \\ &= 528 \text{ J.} \end{aligned}$$

Example 2:

A librarian displaces a book from an upper shelf to a lower one. If the vertical distance between the two shelves is 0.5 meters and the weight of the book is 5 Newtons . How much work is done by the librarian?

Solution:

In order to be able to lift the book and move it to its new position, the librarian must exert a force that is at least equal to the weight of the book. In addition, since the displacement is a vector quantity, then the direction must be taken into account. So,

$$d = -0.5 \text{ meters}.$$

Thus

$$\begin{aligned} W &= Fd \\ &= (5)(-0.5) \\ &= -2.5 \text{ J}. \end{aligned}$$

Here we say that the work is negative since there is a loss of gravitational potential energy rather than a gain in energy. If the book is lifted to a higher shelf, then the work is positive, since there will be a gain in the gravitational potential energy.

Example 3:

A bucket has an empty weight of 23 N. It is filled with sand of weight 80 N and attached to a rope of weight 5.1 N/m. Then it is lifted from the floor at a constant rate to a height 32 meters above the floor. While in flight, the bucket leaks sand grains at a constant rate, and by the time it reaches the top no sand is left in the bucket. Find the work done:

1. by lifting the empty bucket;
2. by lifting the sand alone;
3. by lifting the rope alone;
4. by the lifting the bucket, the sand, and the rope together.

Solution:

1. *The empty bucket.* Since the bucket's weight is constant, the worker must exert a force that is equal to the weight of the empty bucket. Thus

$$\begin{aligned} W &= Fd \\ &= (23)(+32) \\ &= 736 \text{ J}. \end{aligned}$$

2. *The sand alone.* The weight of the sand is decreasing at a constant rate from 80 N to 0 N over the 32 − meter lift. When the bucket is at x meters above the floor, the sand weighs

$$\begin{aligned} F(x) &= [\text{original weight of sand}][\text{proportion left at elevation } x] \\ &= 80\left(\frac{32-x}{32}\right) \\ &= 80\left(1-\frac{x}{32}\right) \\ &= 80 - 2.5x \text{ N}. \end{aligned}$$

The graph of $F(x) = 80 - 2.5x$ represents the variation of the force with height x (Figure 23). The work done corresponds to computing the area under the force graph.

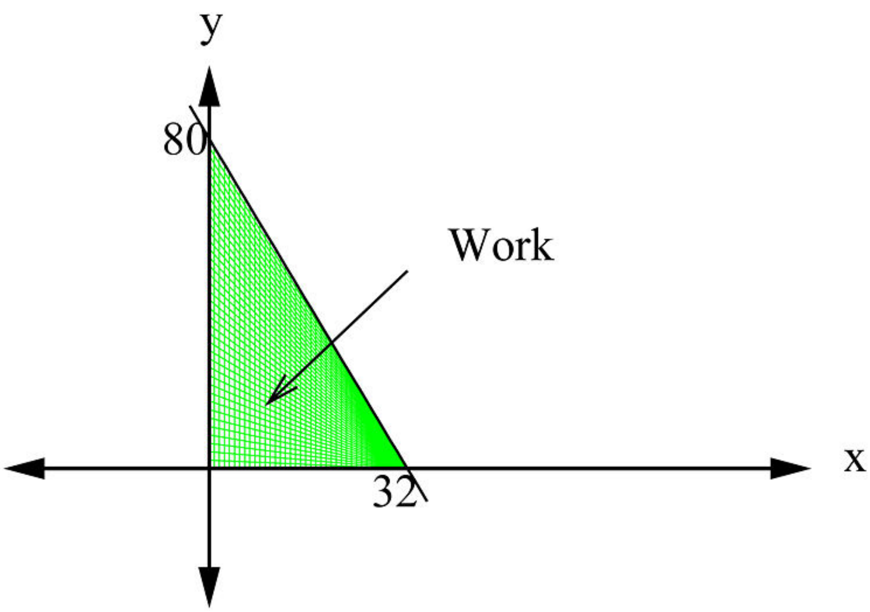


Figure 23

Thus the work done is

$$\begin{aligned} W &= \int_a^b F(x)dx \\ &= \int_0^{32} [80 - 2.5x]dx \\ &= \left[80x - \frac{2.5}{2}x^2\right]_0^{32} \\ &= 1280 \text{ J.} \end{aligned}$$

3. *The rope alone.* Since the weight of the rope is 5.1 N/m and the height is 32 meters , the total weight of the rope from the floor to a height of 32 meters is

$$(5.1)(32) = 163.2 \text{ N.}$$

But since the worker is constantly pulling the rope, the rope's length is decreasing at a constant rate and thus its weight is also decreasing as the bucket being lifted. So at x meters, the $(32 - x)$ meters there remain to be lifted of weight $F(x) = (5.1)(32 - x)$ N. Thus the work done to lift the weight of the rope is

$$\begin{aligned} W &= \int_0^{32} F(x)dx = \int_0^{35} (5.1)(32 - x)dx \\ W &= (5.1)\left[32x - \frac{x^2}{2}\right]_0^{32} \\ &= 2611.2 \text{ J.} \end{aligned}$$

4. *The bucket, the sand, and the rope together*. Here we are asked to sum all the work done on the empty bucket, the sand, and the rope. Thus

$$W_{total} = 736 + 1280 + 2611.2 = 4627.2 \text{ J}.$$

Fluid Statics: Pressure

You have probably studied that ***pressure*** is defined as the force per area

$$P = \frac{F}{A},$$

which has the units of Pascals (Pa) or Newtons per meter squared, $\text{Pa} = \text{N}/\text{m}^2$. In the study of fluids, such as water pressure on a dam or water pressure in the ocean at a depth h, another equivalent formula can be used. It is called the ***liquid pressure*** P at depth h:

$$P = wh.$$

where w is the *weight density*, which is the weight of the column of water per unit volume. For example, if you are diving in a pool, the pressure of the water on your body can be measured by calculating the total weight that the column of water is exerting on you times your depth. Another way to express this formula, the weight density w, is defined as

$$w = \rho g,$$

where ρ is the density of the fluid and g is the acceleration due to gravity (which is $g = 9.8 \text{ m}/\text{sec}^2$ on Earth). The pressure then can be written as

$$P = wh = \rho gh.$$

Example 4:

What is the total pressure experienced by a diver in a swimming pool at a depth of 2 meters ?

Solution

First we calculate the fluid pressure the water exerts on the diver at a depth of 2 meters :

$$P = \rho gh.$$

The density of water is $\rho = 1000 \text{ kg}/\text{m}^3$, thus

$$\begin{aligned} P &= (1000)(9.8)(2) \\ &= 19600 \text{ Pa}. \end{aligned}$$

The total pressure on the diver is the pressure due to the water plus the atmospheric pressure. If we assume that the diver is located at sea-level, then the atmospheric pressure at sea level is about 10^5 Pa . Thus the total pressure on the diver is

$$\begin{aligned} P_{total} &= P_{water} + P_{atm} \\ &= 19600 + 10^5 \\ &= 119600 \\ &= 1.196 \times 10^5 \text{ Pa.} \end{aligned}$$

Example 5:

What is the fluid pressure (excluding the air pressure) and force on the top of a flat circular plate of radius 3 meters that is submerged horizontally at a depth of 10 meters ?

Solution:

The density of water is $\rho = 1000 \text{ kg/m}^3$. Then

$$\begin{aligned} P &= \rho g h \\ &= (1000)(9.8)(10) \\ &= 98000 \text{ Pa.} \end{aligned}$$

Since the force is $F = PA$, then

$$\begin{aligned} F &= PA \\ &= P \cdot \pi r^2 \\ &= (98000)(\pi)(3)^2 \\ &= 2.77 \times 10^6 \text{ N.} \end{aligned}$$

As you can see, it is easy to calculate the fluid force on a horizontal surface because each point on the surface is at the same depth. The problem becomes a little complicated when we want to calculate the fluid force or pressure if the surface is vertical. In this situation, the pressure is not constant at every point because the depth is not constant at each point. To find the fluid force or pressure on a vertical surface we must use calculus.

The Fluid Force on a Vertical Surface

Suppose a flat surface is submerged vertically in a fluid of weight density w and the submerged portion of the surface extends from $x = a$ to $x = b$ along the vertical $x-$axis, whose positive direction is taken as downward. If $L(x)$ is the width of the surface and $h(x)$ is the depth of point x, then the ***fluid force*** F is defined as

$$F = \int_a^b wh(x)L(x)dx.$$

Example 6:

A perfect example of a vertical surface is the face of a dam. We can picture it as a rectangle of a certain height and certain width. Let the height of the dam be 100 meters and of width of 300 meters. Find the total fluid force exerted on the face if the top of the dam is level with the water surface (Figure 24).

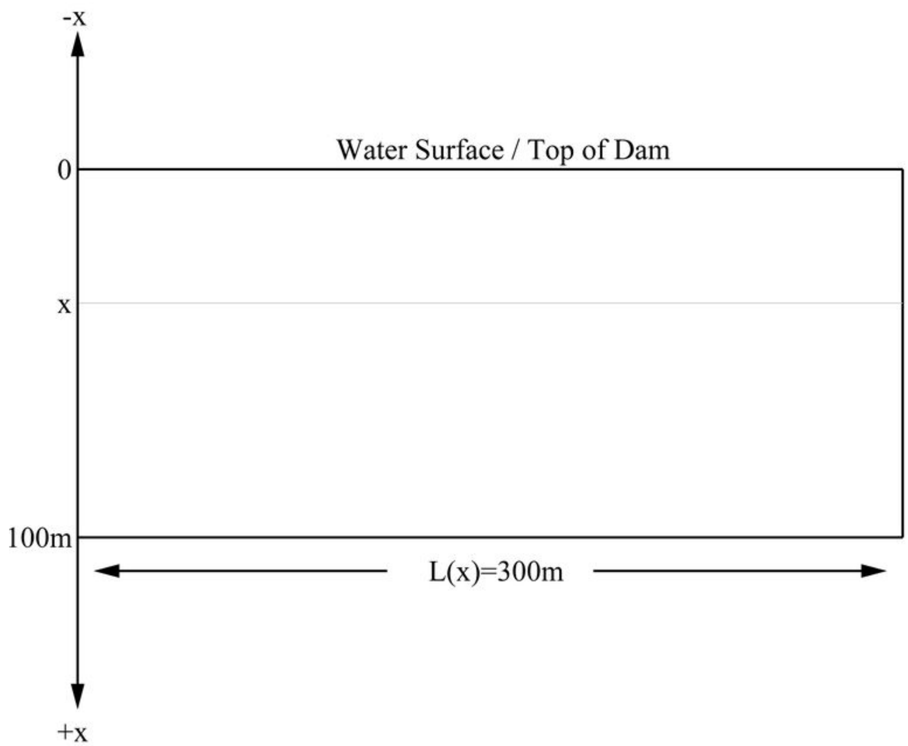


Figure 24

Solution:

Let $x =$ the depth of the water. At an arbitrary point x on the dam, the width of the dam is $L(x) = 300$ m and the depth is $h(x) = xm$. The weight density of water is

$$\begin{aligned} w_{water} &= \rho g \\ &= (1000)(9.8) \\ &= 9800 \text{ N/m}^2. \end{aligned}$$

Using the fluid force formula above,

$$\begin{aligned} F &= \int_a^b wh(x)L(x)dx \\ &= \int_0^{100} (9800)(x)(300)dx \\ &= 2.94 \times 10^6 \int_0^{100} xdx \\ &= 2.94 \times 10^6 \left[\frac{x^2}{2}\right]_0^{100} \\ &= 1.47 \times 10^{10} \text{ N}. \end{aligned}$$

Normal Probabilities

If you were told by the postal service that you will receive the package that you have been waiting for sometime tomorrow, what is the probability that you will receive it sometime between 3:00 PM and 5:00 PM if you know that the postal service's hours of operations are between 7:00 AM to 6:00 PM?

If the hours of operations are between 7 AM to 6 PM, this means they operate for a total of 11 hours. The interval between 3 PM and 5 PM is 2 hours, and thus the probability that your package will arrive is

$$\begin{aligned} P &= \frac{2 \text{ hours}}{11 \text{ hours}} = 0.182 \\ &= 18.2\% \end{aligned}$$

So there is a probability of 18.2% that the postal service will deliver your package sometime between the hours of 3 PM and 5 PM (or during any $2-$hour interval). That is easy enough. However, mathematically, the situation is not that simple. The $11-$hour interval and the $2-$hour interval contain an infinite number of times. So how can one infinity over another infinity produce a probability of 18.2%? To resolve this issue, we represent the total probability of the $11-$hour interval as a rectangle of area 1 (Figure 25).

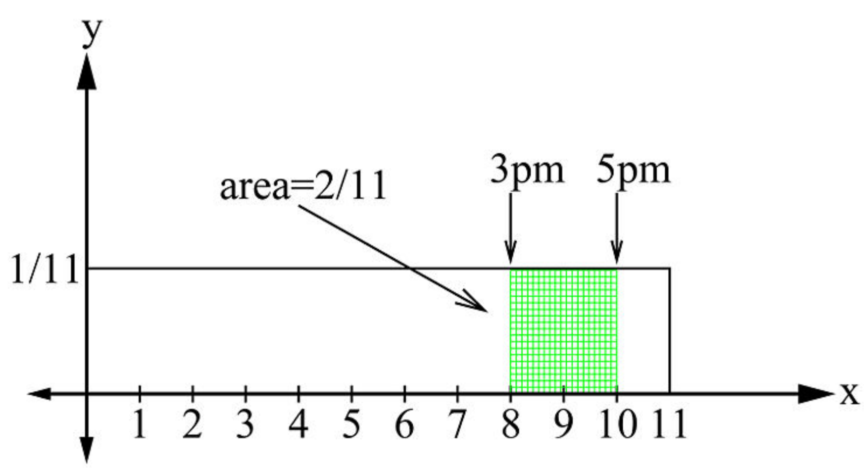


Figure 25

Looking at the $2-$hour interval, we can see that it is equal to $\frac{2}{11}$ of the total rectangular area 1. This is why it is convenient to represent probabilities as areas. But since areas can be defined by definite integrals, we can also define the probability associated with an interval $[a,b]$ by the definite integral

$$P = \int_a^b f(x)dx,$$

where $f(x)$ is called the ***probability density function*** (pdf). One of the most useful probability density functions is the ***normal curve*** or the ***Gaussian curve*** (and sometimes the ***bell curve***) (Figure 26). This function enables us to describe an entire population based on statistical measurements taken from a small sample of the population. The only measurements needed are the mean (μ) and the standard deviation (σ). Once those two numbers are known, we can easily find the normal curve by using the following formula.

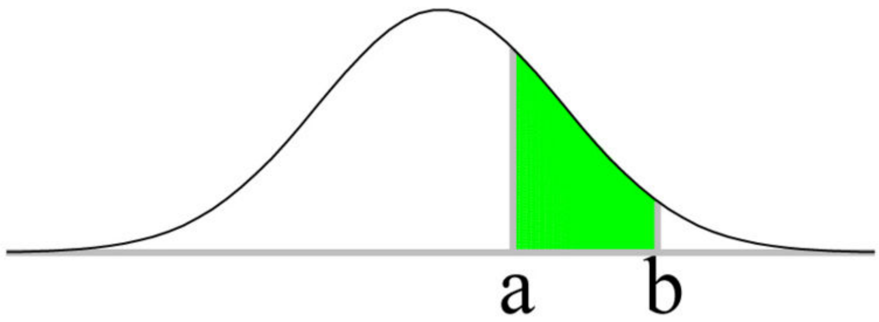


Figure 26

The Normal Probability Density Function

The Gaussian curve for a population with mean μ and standard deviation σ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)},$$

where the factor $1/(\sigma\sqrt{2\pi})$ is called the *normalization constant.* It is needed to make the probability over the entire space equal to 1. That is,

$$P(-\infty < x < \infty) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} = 1.$$

Example 7:

Suppose that boxes containing 100 tea bags have a mean weight of 10.2 ounces each and a standard deviation of 0.1 ounce.

1. What percentage of all the boxes is expected to weigh between 10 and 10.5 ounces ?
2. What is the probability that a box weighs less than 10 ounces ?
3. What is the probability that a box will weigh exactly 10 ounces ?

Solution:

1. Using the normal probability density function,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

Substituting for $\mu = 10.2$ and $\sigma = 0.1$, we get

$$f(x) = \frac{1}{(0.1)\sqrt{2\pi}} e^{-(x-10.2)^2/(2(0.1)^2)}.$$

The percentage of all the tea boxes that are expected to weight between 10 and 10.5 ounces can be calculated as

$$P(10 \leq x \leq 10.5) = \int_{10}^{10.5} \frac{1}{(0.1)\sqrt{2\pi}} e^{-(x-10.2)^2/(2(0.1)^2)} dx.$$

The integral of e^{x^2} does not have an elementary anti-derivative and therefore cannot be evaluated by standard techniques. However, we can use numerical techniques, such as The Simpson's Rule or The Trapezoid Rule, to find an approximate (but very accurate) value. Using the programing feature of a scientific calculator or, mathematical software, we eventually get

$$\int_{10}^{10.5} \frac{1}{(0.1)\sqrt{2\pi}} e^{-(x-10.2)^2/(2(0.1)^2)} dx \approx 0.976.$$

That is,

$$P(10 \leq x \leq 10.5) \approx 97.6\%.$$

Technology Note: To make this computation with a graphing calculator of the TI-83/84 family, do the following:

- From the **[DISTR]** menu (Figure 27) select option 2, which puts the phrase "normalcdf" in the home screen. Add lower bound, upper bound, mean, standard deviation, separated by commas, close the parentheses, and press **[ENTER]**. The result is shown in Figure 28.

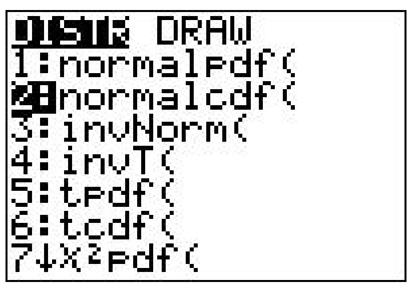


Figure 27

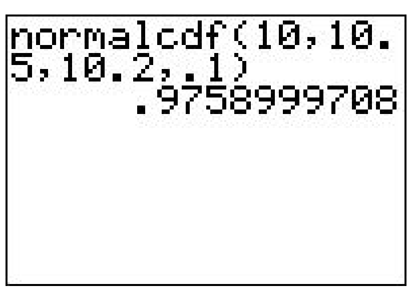


Figure 28

2. For the probability that a box weighs less than 10.2 ounces, we use the area under the curve to the left of $x = 10.2$. Since the value of $f(9)$ is very small (less than a billionth),

$$\begin{aligned} f(9) &= \frac{1}{(0.1)\sqrt{2\pi}} e^{-(9-10.2)^2/(2(0.1)^2)} dx \\ &= 1.35 \times 10^{-32}, \end{aligned}$$

getting the area between 9 and 10 will yield a fairly good answer. Integrating numerically, we get

$$\begin{aligned} P(9 \le x \le 10) &= \int_9^{10} \frac{1}{(0.1)\sqrt{2\pi}} e^{-(x-10.2)^2/(2(0.1)^2)} dx \\ P(9 \le x \le 10.2) &\approx 0.02275 \\ &= 2.28\%, \end{aligned}$$

which says that we would expect 2.28% of the boxes to weigh less than 10 ounces.

3. Theoretically the probability here will be exactly zero because we will be integrating from 10 to 10, which is zero. However, since all scales have some error (call it ε), practically we would find the probability that the weight falls between $10 - \varepsilon$ and $10 + \varepsilon$.

Example 8:

An Intelligence Quotient or IQ is a score derived from different standardized tests attempting to measure the level of intelligence of an adult human being. The average score of the test is 100 and the standard deviation is 15.

1. What is the percentage of the population that has a score between 85 and 115?
2. What percentage of the population has a score above 140?

Solution:

1. Using the normal probability density function,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)},$$

and substituting $\mu = 100$ and $\sigma = 15$,

$$f(x) = \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2(15)^2)}.$$

The percentage of the population that has a score between 85 and 115 is

$$P(85 \le x \le 115) = \int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2(15)^2)}.$$

Again, the integral of e^{-x^2} does not have an elementary anti-derivative and therefore cannot be evaluated. Using the programing feature of a scientific calculator or a mathematical computer software, we get

$$\int_{85}^{115} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2(15)^2)} dx \approx 0.68.$$

That is,

$$P(85 \leq x \leq 115) \approx 68\%.$$

Which says that 68% of the population has an IQ score between 85 and 115.

2. To measure the probability that a person selected randomly will have an IQ score above 140,

$$P(x \geq 140) = \int_{140}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2(15)^2)} dx.$$

This integral is even more difficult to integrate since it is an improper integral. To avoid the messy work, we can argue that since it is extremely rare to meet someone with an IQ score of over 200, we can approximate the integral from 140 to 200. Then

$$P(x \geq 140) \approx \int_{140}^{200} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/(2(15)^2)} dx.$$

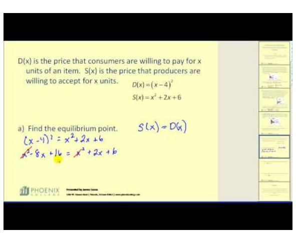
Integrating numerically, we get

$$P(x \geq 140) \approx 0.0039.$$

So the probability of selecting at random a person with an IQ score above 140 is 0.39%. That's about one person in every 250 individuals!

Multimedia Links

For a video presentation of an application of integration involving consumer and producer surplus **(14.0)**, see Math Video Tutorials by James Sousa, Consumer and Producer Surplus (10:22).



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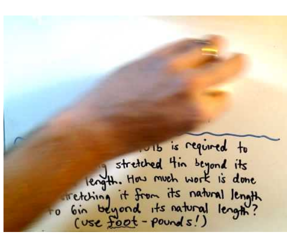
For video presentations of work and Hooke's Law **(14.0)(16.0)**, see Just Math Tutoring, Work and Hooke's Law, Example 1 (5:00)



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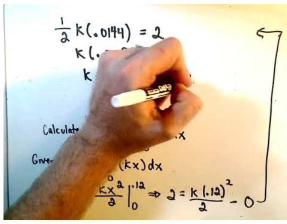
and Just Math Tutoring, Work and Hooke's Law, Example 2 (6:52).



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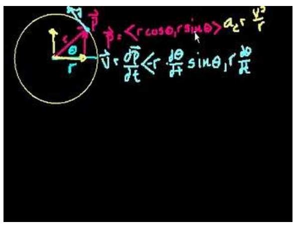
For a video that uses calculus to explain centripetal acceleration **(16.0)**, see Khan Academy, Centripetal Accelerati on (10:14).



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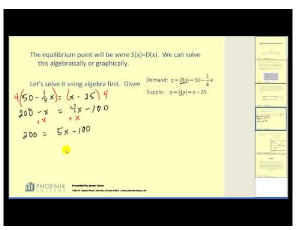

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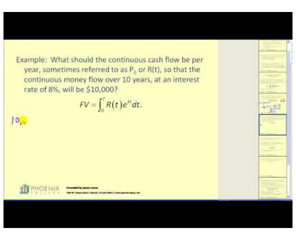

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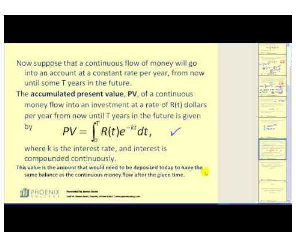

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Review Questions

1. A particle moves along the $x-$axis by a force

$$F(x) = \frac{1}{x^2+1}.$$

If the particle has already moved a distance of 10 meters from the origin, what is the work done by the force?
2. A force of $\cos\left(\frac{\pi x}{2}\right)$ acts on an object when it is x meters away from the origin. How much work is done by this force in moving the object from $x = 1$ to $x = 5$ meters?
3. In physics, if the force on an object varies with distance then work done by the force is defined as (see Example 5.15)

$$W = \int_a^b F(r)dr.$$

That is, the work done corresponds to computing the area under the force graph. For example, the strength of the gravitational field varies with the distance r from the Earth's center. If a satellite of mass m is to be launched into space, then the force experienced by the satellite during and after launch is

$$F(r) = G\frac{mM}{r^2},$$

where $M = 6\times 10^{24}$ kg is the mass of the Earth and $G = 6.67\times 10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2}$ is the Universal Gravitational Constant. If the mass of the satellite is 1000 kg and we wish to lift it to an altitude of $35,780$ km above the Earth's surface, how much work is needed to lift it? (Radius of Earth is 6370 km.)
4. *Hook's Law* states that when a spring is stretched x units beyond its natural length it pulls back with a force

$$F(x) = kx,$$

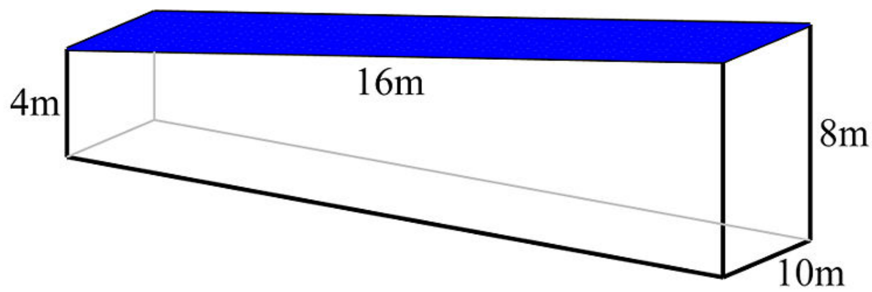
where k is called the *spring constant* or the *stiffness* constant. To calculate the work required to stretch the spring a length x we use

$$W = \int_a^b F(x)dx,$$

where a is the initial displacement of the spring ($a = 0$ if the spring is initially unstretched) and b is the final displacement. A force of 5 N is exerted on a spring and stretches it 1 m beyond its natural length.

   a. Find the spring constant k.
   b. How much work is required to stretch the spring 1.8 m beyond its natural length?

5. When a force of 30 N is applied to a spring, it stretches it from a length of 12 cm to 15 cm. How much work will be done in stretching the spring from 12 cm to 20 cm? (Hint: read the first part of problem #4 above.)
6. A flat surface is submerged vertically in a fluid of weight density w. If the weight density w is doubled, is the force on the plate also doubled? Explain.
7. The bottom of a rectangular swimming pool, whose bottom is an inclined plane, is shown below. Calculate the fluid force on the bottom of the pool when it is filled completely with water.



8. Suppose $f(x)$ is the probability density function for the lifetime of a manufacturer's light bulb, where x is measured in hours. Explain the meaning of each integral.
   a. $\int_{1000}^{5000} f(x)dx$
   b. $\int_{3000}^{\infty} f(x)dx$
9. The length of time a customer spends waiting until his/her entree is served at a certain restaurant is modeled by an exponential density function with an average time of 8 minutes.
   a. What is the probability that a customer is served in the first 3 minutes?
   b. What is the probability that a customer has to wait more than 10 minutes?
10. The average height of an adult female in Los Angeles is 63.4 inches (5 feet3.4 inches) with a standard deviation of 3.2 inches.
    a. What is the probability that a female's height is less than 63.4 inches?
    b. What is the probability that a female's height is between 63 and 65 inches?
    c. What is the probability that a female's height is more than 6 feet?
    d. What is the probability that a female's height is exactly 5 feet?

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9730 .

CHAPTER 6 Transcendental Functions

Chapter Outline

6.1 Inverse Functions

Functions such as logarithms, exponential functions, and trigonometric functions are examples of ***transcendental functions.*** If a function is transcendental, it cannot be expressed as a polynomial or rational function. That is, it is not an *algebraic function.* In this chapter, we will begin by developing the concept of an inverse of a function and how it is linked to its original numerically, algebraically, and graphically. Later, we will take each type of elementary transcendental function—logarithmic, exponential, and trigonometric—individually and see the connection between them and their respective inverses, derivatives, and integrals.

Learning Objectives

A student will be able to:

- Understand the basic properties of the inverse of a function and how to find it.
- Understand how a function and its inverse are represented graphically.
- Know the conditions of invertabilty of a function.

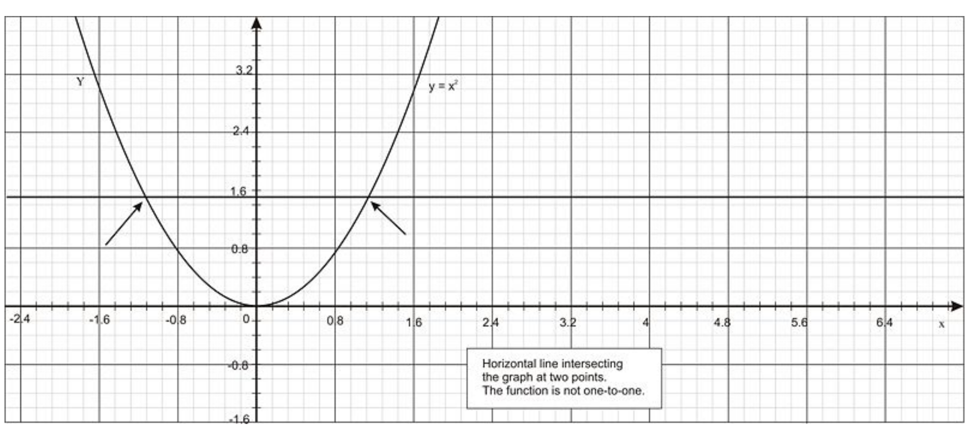
One-to-One Functions

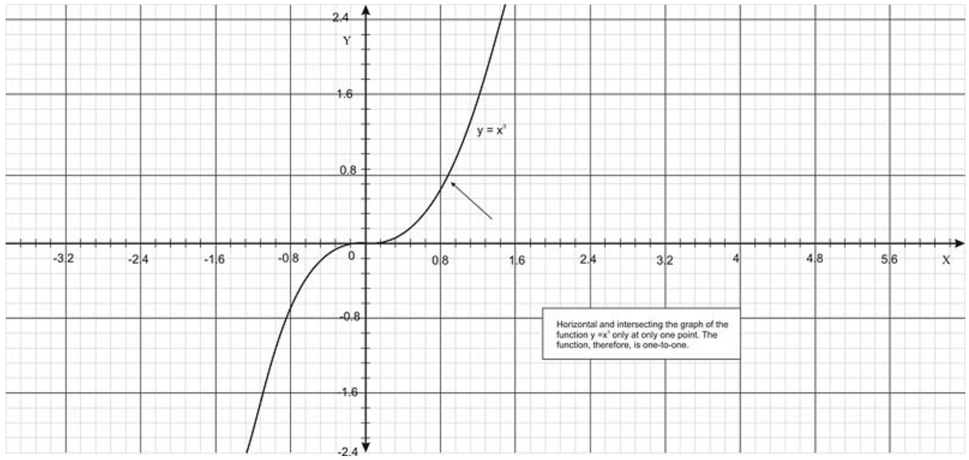
A function, as you know from your previous mathematics background, is a rule that assigns a single value in its range to each point in its domain. In other words, for each output number, there is one or more input numbers. However, a function never produces more than a single output for one input. A function is said to be a ***one-to-one*** function if each output is associated with only one single input. For example, $f(x) = x^2$ assigns the output 9 for both 3 and -3, and thus it is not a *one-to-one* function.

One-to-One Function

The function $f(x)$ is one-to-one in a domain D if $f(a) \neq f(b)$, whenever $a \neq b$.

There is an easy method to check if a function is one-to-one: draw a horizontal line across the graph. If the line intersects at only one point on the graph, then the function is one-to-one; otherwise, it is not. Notice in the figure below that the graph of $y = x^2$ is not one-to-one since the horizontal line intersects the graph more than once. But the function $y = x^3$ is a one-to-one function because the graph meets the horizontal line only once.



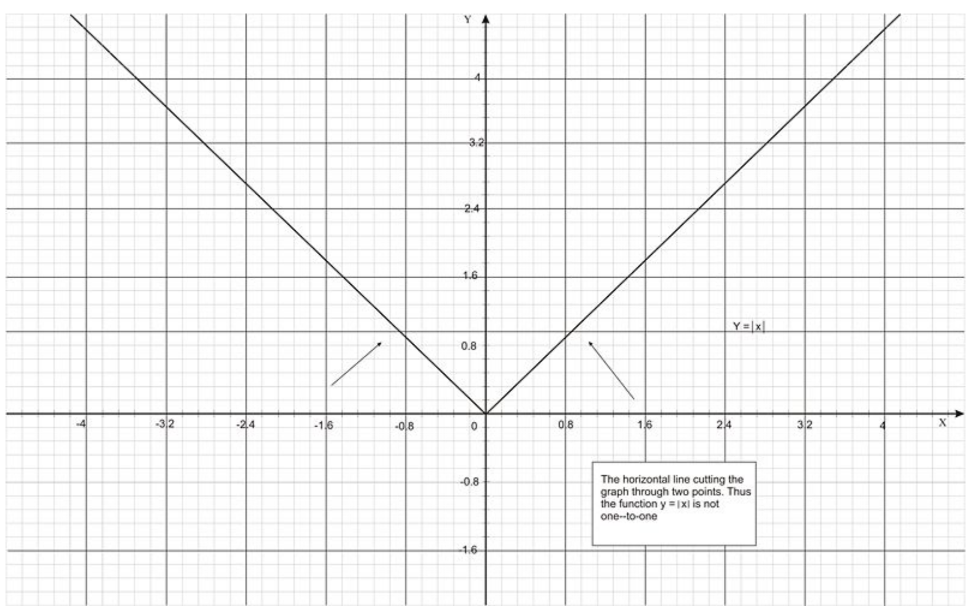


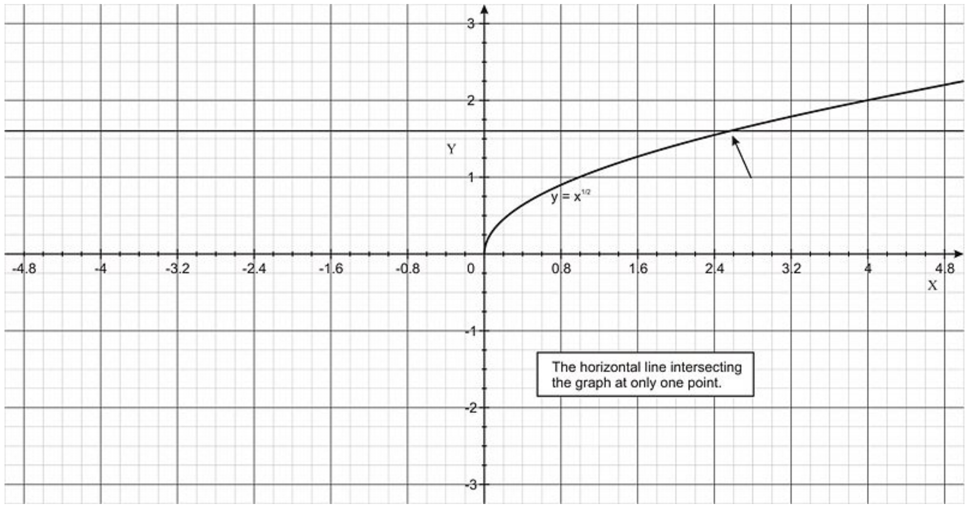
Example 1:

Determine whether the functions are one-to-one: (a) $f(x) = |x|$ (b) $h(x) = x^{1/2}$.

Solution:

It is best to graph both functions and draw on each a horizontal line. As you can see from the graphs, $f(x) = |x|$ is not one-to-one since the horizontal line intersects it at two points. The function $h(x) = x^{1/2}$, however, is indeed one-to-one since only one point is intersected by the horizontal line.





The Inverse of a Function

We discussed above the condition for a one-to-one function: for each output, there is only one input. A one-to-one function can be reversed in such a way that the input of the function becomes the output and the output becomes an input. This reverse of the original function is called the *inverse* of the function. If f^{-1} is an inverse of a function f, then $f^{-1} \circ f = x$. For example, the two functions $f(x) = 2x + 3$ and $h(x) = \frac{x-3}{2}$ are inverses of each other since

$$f \circ h = f(h(x)) = 2\left[\frac{x-3}{2}\right] + 3 = x - 3 + 3 = x,$$
$$h \circ f = h(f(x)) = \frac{(2x+3)-3}{2} = \frac{2x}{x} = x.$$

Thus

$$f \circ h = h \circ f = x,$$

and f and h are inverses of each other.

Note: In general, $f^{-1} \neq \frac{1}{f}$.

When is a function invertable?

It is interesting to note that if a function $f(x)$ is always increasing or always decreasing over its domain, then a horizontal line will cut through this graph at one point only. Then f in this case is a one-to-one function and thus has an inverse. So if we can find a way to prove that a function is constantly increasing or decreasing, then it is ***invertable*** or ***monotonic***. From previous chapters, you have learned that if $f'(x) > 0$ then f must be increasing and if $f'(x) < 0$ then f must be decreasing.

To summarize, a function has an inverse if it is one-to-one in its domain or if its derivative is either $f'(x) > 0$ or $f'(x) < 0$.

Example 2:

Given the polynomial function $f(x) = 3x^5 + 2x + 1$, show that it is invertable (has an inverse).

Solution:

Taking the derivative, we find that $f'(x) = 15x^4 + 2 > 0$ for all x. We conclude that $f(x)$ is one-to-one and invertable. Keep in mind that it may not be easy to find the inverse of $f(x) = 3x^5 + 2x + 1$ (try it!), but we still know that it is indeed invertable.

How to find the inverse of a one-to-one function:

To find the inverse of a one-to-one function, simply solve for x in terms of y and then interchange x and y. The resulting formula is the inverse $y = f^{-1}(x)$.

Example 3:

Find the inverse of $f(x) = \sqrt{4x+1}$.

Solution:

From the discussion above, we can find the inverse by first solving for x in $y = \sqrt{4x+1}$.

$$\begin{aligned} y &= \sqrt{4x+1}, \\ y^2 &= 4x+1, \\ x &= \frac{y^2-1}{4}. \end{aligned}$$

Interchanging $x \longleftrightarrow y$,

$$y = \frac{x^2-1}{4}.$$

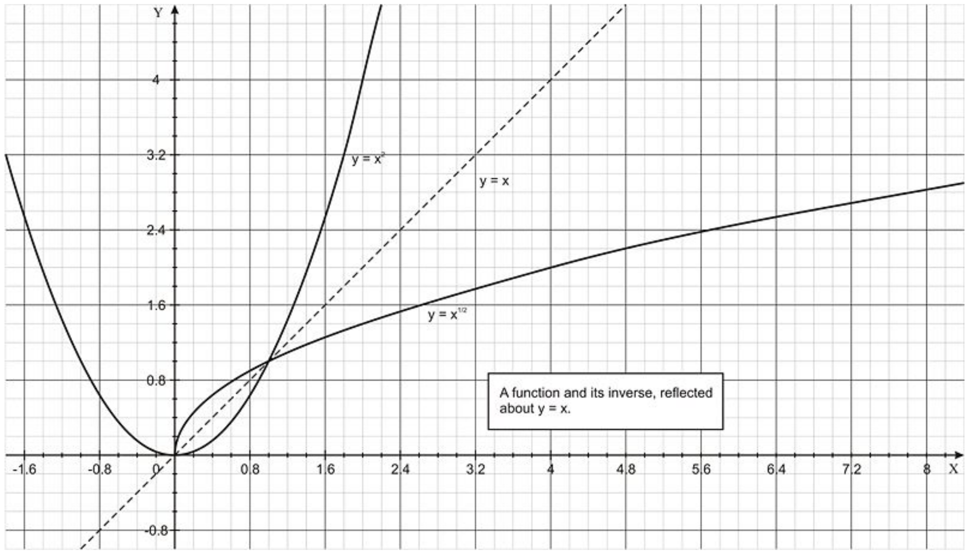
Replacing $y = f^{-1}(x)$,

$$f^{-1}(x) = \frac{x^2-1}{4}$$

which is the inverse of the original function $f(x) = \sqrt{4x+1}$.

Graphs of Inverse Functions

What is the relationship between the graphs of f and f^{-1}? If the point (a,b) is on the graph of $f(x)$, then from the definition of the inverse, the point (b,a) is on the graph of $f^{-1}(x)$. In other words, when we reverse the coordinates of a point on the graph of $f(x)$ we automatically get a point on the graph of $f^{-1}(x)$. We conclude that $f(x)$ and $f^{-1}(x)$ are ***reflections*** of one another about the line $y = x$. That is, each is a mirror image of the other about the line $y = x$. The figure below shows an example of $y = x^2$ and, when the domain is restricted, its inverse $y = \sqrt{x}$ and how they are reflected about $y = x$.



It is important to note that for the function $f(x) = x^2$ to have an inverse, we must restrict its domain to $0 \leq x < \infty$, since that is the domain in which the function is increasing.

Continuity and Differentiability of Inverse Functions

Since the graph of a one-to-one function and its inverse are reflections of one another about the line $y = x$, it would be safe to say that if the function f has no breaks (no discontinuities) then f^{-1} will not have breaks either. This implies that if f is continuous on the domain D, then its inverse f^{-1} is continuous on the range R of f. For example, if $f(x) = \sqrt{x}$, then its domain is $x \geq 0$ and its range is $y \geq 0$. This means that $f(x)$ is continuous for all $x \geq 0$. The inverse of $f(x)$ is $f^{-1}(x) = x^2$, where its domain is all $x > 0$ and its range is $y \geq 0$. We conclude that if f is a function with domain D and range R and it is continuous and one-to-one on D, then its inverse f^{-1} is continuous and one-to-one on the range R of f.

Suppose that f has a domain D and a range R. If f is differentiable and one-to-one on D, then its inverse f^{-1} is differentiable at any value x in R for which $f'(f^{-1}(x)) \neq 0$ and

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}.$$

The formula above can be written in a form that is easier to remember:

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

In addition, if f on its domain is either $f'(x) > 0$ or $f'(x) < 0$, then f has an inverse function f^{-1} and f^{-1} is differentiable at all values of x in the range of f. In this case, f^{-1} is given by the formula above. The example below illustrate this important theorem.

Example 4:

In Example 3, we were given the polynomial function $f(x) = 3x^5 + 2x + 1$ and we showed that it is invertable. Show that it is differentiable and find the derivative of its inverse.

Solution:

Since $f'(x) = 15x^4 + 2 > 0$ for all $x \in R, f^{-1}(x)$ is differentiable at all values of x. To find the derivative of f^{-1}, if we let $x = f(y)$, then

$$x = f(y) = 3y^5 + 2y + 1.$$

So

$$\frac{dx}{dy} = 15y^4 + 2$$

and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{15y^4 + 2}.$$

Since we are unable to solve for y in terms of x, we leave the answer above in terms of y. Another way of solving the problem is to use Implicit Differentiation:

Since

$$x = 3y^5 + 2y + 1,$$

differentiating implicitly,

$$\begin{aligned} \frac{d}{dx}[x] &= \frac{d}{dx}[3y^5 + 2y + 1], \\ 1 &= (15y^4 + 2)\frac{dy}{dx}. \end{aligned}$$

Solving for $\frac{dy}{dx}$ we finally obtain

$$\frac{dy}{dx} = \frac{1}{15y^4 + 2},$$

which is the same result.

Review Questions

In problems #1 - 3, find the inverse function of f and verify that $f \circ f^{-1} = f^{-1} \circ f = x$.

1. $f(x) = 3x + 1$
2. $\sqrt[3]{x}$

3. $f(x) = \frac{x-1}{3}$

In problems #4 - 6, use the horizontal line test to verify whether the following functions have inverse.

4. $h(x) = \frac{4-x}{6}$
5. $g(x) = |x+4| - |x-4|$
6. $f(x) = -2x\sqrt{16-x^2}$

In problems #7 - 8, use the functions $f(x) = x+4$ and $g(x) = 2x-5$ to find the specified functions.

7. $g^{-1} \circ f^{-1}$
8. $(f \circ g)^{-1}$

In problems #9 - 10, show that f is monotonic (invertable) on the given interval (and therefore has an inverse.)

9. $f(x) = (x-5)^2, [5, \infty)$
10. $f(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

6.2 Exponential and Logarithmic Functions

Learning Objectives

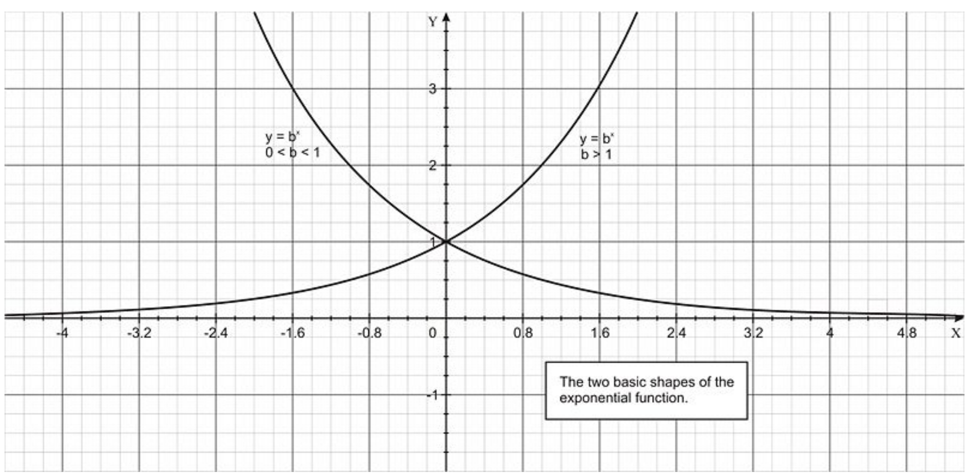
A student will be able to:

- Understand and use the basic definitions of exponential and logarithmic functions and how they are related algebraically.
- Distinguish between an exponential and logarithmic functions graphically.

A Quick Algebraic Review of Exponential and Logarithmic Functions

Exponential Functions

Recall from algebra that an exponential function is a function that has a constant base and a variable exponent. A function of the form $f(x) = b^x$ where b is a constant and $b > 0$ and $b \neq 1$ is called an exponential function with base b. Some examples are $f(x) = 2^x, f(x) = \left(\frac{1}{2}\right)^x$, and $f(x) = e^x$. All exponential functions are continuous and their graph is one of the two basic shapes, depending on whether $0 < b < 1$ or $b > 1$. The graph below shows the two basic shapes:



Logarithmic Functions

Recall from your previous courses in algebra that a logarithm is an exponent. If the base $b > 0$ and $b \neq 1$, then for any value of $x > 0$, the logarithm to the base b of the value of x is denoted by

$$y = \log_b x.$$

This is equivalent to the exponential form

$$b^y = x.$$

For example, the following table shows the logarithmic forms in the first row and the corresponding exponential forms in the second row.

Logarithmic Form $\rightarrow$	$\log_2 16 = 4$	$\log_5 \frac{1}{25} = -2$	$\log_{10} 100 = 2$	$\log_e e = 1$
Exponential Form $\rightarrow$	$2^4 = 16$	$5^{-2} = \frac{1}{25}$	$10^2 = 100$	$e^1 = e$

Historically, logarithms with base of 10 were very popular. They are called the common logarithms. Recently the base 2 has been gaining popularity due to its considerable role in the field of computer science and the associated binary number system. However, the most widely used base in applications is the natural logarithm, which has an irrational base denoted by e, in honor of the famous mathematician Leonhard Euler. This irrational constant is $e \approx 2.718281$. Formally, it is defined as the limit of $(1+x)^{1/x}$ as x approaches zero. That is,

$$\lim_{x \to 0}(1+x)^{1/x} = e.$$

We denote the natural logarithm of x by $\ln x$ rather than $\log_e x$. So keep in mind, that $\ln x$ is the power to which e must be raised to produce x. That is, the following two expressions are equivalent:

$$y = \ln x \Longleftrightarrow x = e^y$$

The table below shows this operation.

Natural Logarithm ln	$\ln 2 = 0.693$	$\ln 1 = 0$	$\ln e = 1$	$\ln e^3 = 3$
Equivalent Exponential Form	$e^{0.693} = 2$	$e^0 = 1$	$e^1 = e$	$e^3 = e^3$

A Comparison between Logarithmic Functions and Exponential Functions

Looking at the two graphs of exponential functions above, we notice that both pass the horizontal line test. This means that an exponential function is a one-to-one function and thus has an inverse. To find a formula for this inverse, we start with the exponential function

$$y = b^x.$$

Interchanging x and y,

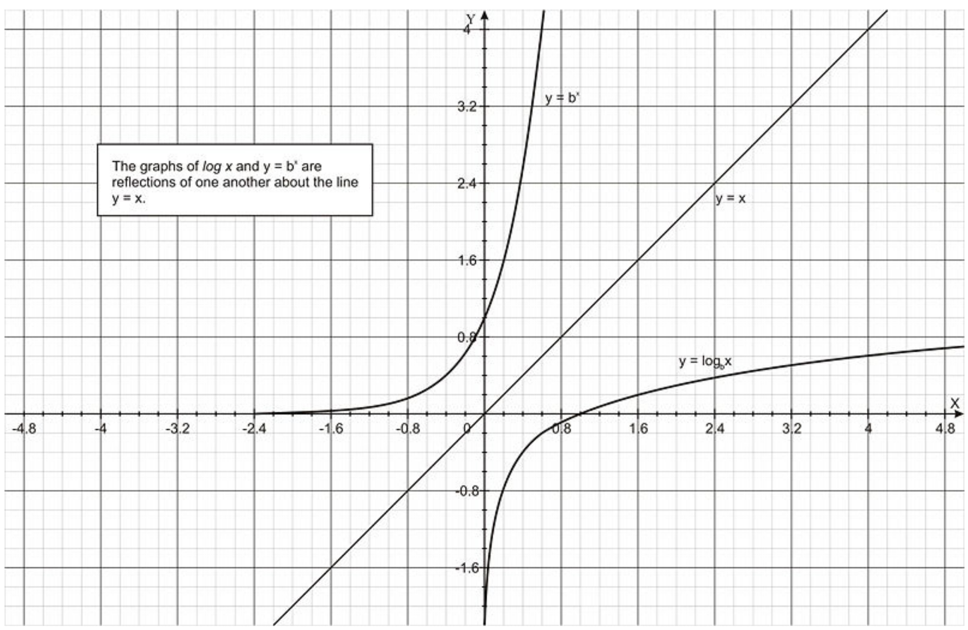
$$x = b^y.$$

Projecting the logarithm to the base b on both sides,

$$\begin{aligned}\log_b x &= \log_b b^y \\ &= y \log_b b \\ &= y(1) \\ &= y.\end{aligned}$$

Thus $y = f^{-1}(x) = \log_b x$ is the inverse of $y = f(x) = b^x$.

This implies that the graphs of f and f^{-1} are reflections of one another about the line $y = x$. The figure below shows this relationship.



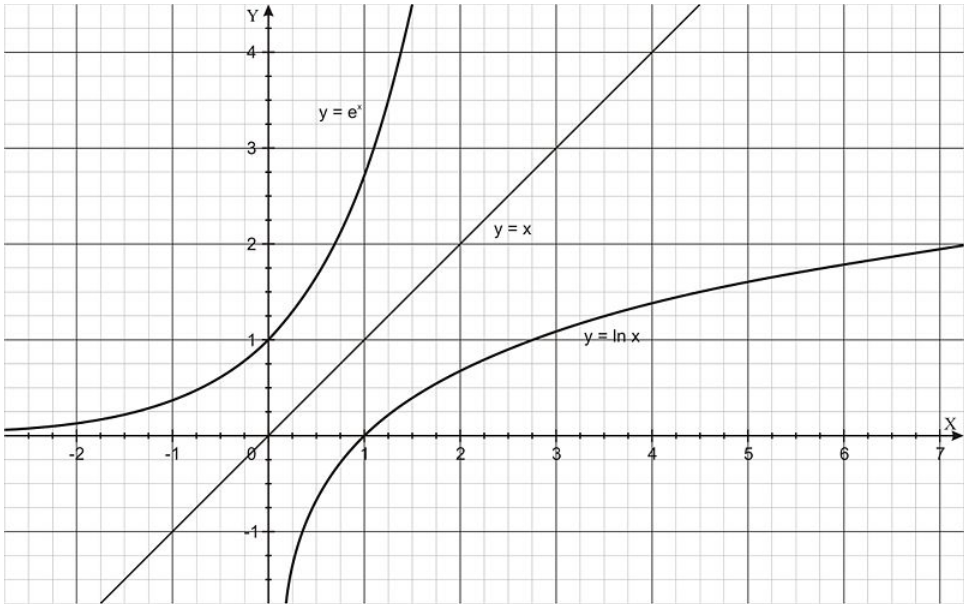
Similarly, in the special case when the base $b = e$, the two equations above take the forms

$$y = f(x) = e^x$$

and

$$f^{-1}(x) = \ln x.$$

The graph below shows this relationship:



Before we move to the calculus of exponential and logarithmic functions, here is a summary of the two important relationships that we have just discussed:

- The function $y = b^x$ is equivalent to $x = \log_b y$ if $y > 0$ and $x \in R$.
- The function $y = e^x$ is equivalent to $x = \ln y$ if $y > 0$ and $x \in R$.

You should also recall the following important properties about logarithms:

- $\log_b vw = \log_b v + \log_b w$
- $\log_b \frac{v}{w} = \log_b v - \log_b w$
- $\log_b w^n = n \log_b w$
- To express a logarithm with base in terms of the natural logarithm: $\log_b w = \frac{\ln w}{\ln b}$
- To express a logarithm with base b in terms of another base a: $\log_b w = \frac{\log_a w}{\log_a b}$

Review Questions

Solve for x.

1. $6^x = \frac{1}{216}$
2. $e^x = 3$
3. $\log_2 z = 3$
4. $\ln x^2 = 5$
5. $3e^{-5x} = 132$
6. $e^{2x} - 7e^x + 10 = 0$
7. $-4(3)^x = -36$
8. $\ln x - \ln 3 = 2$
9. $y = 5\log_{10}\left(\frac{2}{2-x}\right)$
10. $y = 3e^{-2x/3}$

6.3 Differentiation and Integration of Logarithmic and Exponential Functions

Learning Objectives

A student will be able to:

- Understand and use the rules of differentiation of logarithmic and exponential functions.
- Understand and use the rules of integration of logarithmic and exponential functions.

In this section we will explore the derivatives of logarithmic and exponential functions. We will also see how the derivative of a one-to-one function is related to its inverse.

The Derivative of a Logarithmic Function

Our goal at this point to find an expression for the derivative of the logarithmic function $y = \log_b x$. Recall that the exponential number e is defined as

$$e = \lim_{a \to 0} (1+a)^{1/a}$$

(where we have substituted a for x for convenience). From the definition of the derivative of $f(x)$ that you already studied in Chapter 2,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}.$$

We want to apply this definition to get the derivative to our logarithmic function $y = \log_b x$. Using the definition of the derivative and the rules of logarithms from the Lesson on Exponential and Logarithmic Functions,

$$\begin{aligned}
\frac{d}{dx}[\log_b x] &= \lim_{w\to x} \frac{\log_b w - \log_b x}{w-x} \\
&= \lim_{w\to x} \frac{\log_b (w/x)}{w-x} \\
&= \lim_{w\to x} \left[\frac{1}{w-x} \log_b \left(\frac{w}{x}\right)\right] \\
&= \lim_{w\to x} \left[\frac{1}{w-x} \log_b \left(\frac{x+(w-x)}{x}\right)\right] \\
&= \lim_{w\to x} \left[\frac{1}{w-x} \log_b \left(1+\frac{w-x}{x}\right)\right] \\
&= \lim_{w\to x} \left[\frac{1}{x(w-x)} \log_b \left(1+\frac{w-x}{x}\right)\right] \\
&= \lim_{w\to x} \left[\frac{x}{x(w-x)} \log_b \left(1+\frac{w-x}{x}\right)\right].
\end{aligned}$$

At this stage, let $a=(w-x)/(x)$, the limit of $w\to x$ then becomes $a\to 0$. Substituting, we get

$$\begin{aligned}
&= \lim_{a\to 0} \left[\frac{1}{x}\frac{1}{a} \log_b(1+a)\right] \\
&= \frac{1}{x}\lim_{a\to 0} \left[\frac{1}{a} \log_b(1+a)\right] \\
&= \frac{1}{x}\lim_{a\to 0} \left[\log_b(1+a)^{1/a}\right].
\end{aligned}$$

Inserting the limit,

$$= \frac{1}{x}\log_b \left[\lim_{a\to 0}(1+a)^{1/a}\right].$$

But by the definition $e=\lim_{a\to 0}(1+a)^{1/a}$,

$$\frac{d}{dx}[\log_b x] = \frac{1}{x}\log_b e.$$

From the box above, we can express $\log_b e$ in terms of natural logarithm by the using the formula $\log_b w = \ln w/\ln b$. Then

$$\log_b e = \frac{\ln e}{\ln b} = \frac{1}{\ln b}.$$

Thus we conclude

$$\frac{d}{dx}[\log_b x] = \frac{1}{x\ln b} > 0,$$

and in the special case where $b = e$,

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0.$$

To generalize, if u is a differentiable function of x and if $u(x) > 0$, then the above two equations, after the Chain Rule is applied, will produce the generalized derivative rule for logarithmic functions.

Derivatives of Logarithmic Functions

$$\frac{d}{dx}[\log_b u] = \frac{1}{u \ln b}\frac{du}{dx}$$
$$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}$$

Remark: Students often wonder why the constant e is defined the way it is. The answer is in the derivative of $f(x) = \ln x$. With any other base the derivative of $f(x) = \log_b x$ would be equal $f'(x) = \frac{1}{x \ln b}$, a more complicated expression than $1/x$. Thinking back to another unexpected unit, radians, the derivative of $f(x) = \sin(x)$ is the simple expression $f'(x) = \cos(x)$ only if x is in radians. In degrees, $f'(x) = \frac{\pi}{180}\cos(x)$, which is more cumbersome and harder to remember.

Example 1:

Find the derivative of $y = \ln(2x^2 - 4x + 3)$.

Solution:

Since $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$, for $u = 2x^2 - 4x + 3$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2x^2 - 4x + 3}\frac{d}{dx}\left[2x^2 - 4x + 3\right] \\ &= \frac{1}{2x^2 - 4x + 3}(4x - 4) \\ &= \frac{4(x-1)}{2x^2 - 4x + 3}.\end{aligned}$$

Example 2:

Find $\frac{d}{dx}[\ln(\sin x)]$.

Solution:

$$\begin{aligned}\frac{d}{dx}[\ln(\sin x)] &= \frac{1}{\sin x}\cdot[\cos x] \\ &= \frac{\cos x}{\sin x} \\ &= \cot x.\end{aligned}$$

Example 3:

Find $\frac{d}{dx}\left[\ln(\cos 5x)^3\right]$.

Solution:

Here we use the Chain Rule:

$$\begin{aligned}\frac{d}{dx}\left[\ln(\cos 5x)^3\right] &= \frac{1}{\cos^3 5x}\cdot\left[3(\cos 5x)^2\cdot(-\sin 5x)\cdot(5)\right]\\ &= \frac{1}{\cos^3 5x}\cdot\left[-15\cos^2 5x\cdot\sin 5x\right]\\ &= \frac{-15\sin 5x}{\cos 5x}\\ &= -15\tan 5x.\end{aligned}$$

Example 4:

Find the derivative of $y = x^3 \log_5 2x$.

Solution:

Here we use the Product Rule along with $\frac{d}{dx}[\log_b u] = \frac{1}{u\ln b}\frac{du}{dx}$:

$$\begin{aligned}\frac{d}{dx}[x^3\log_5 2x] &= x^3\cdot\frac{d}{dx}[\log_5 2x] + \frac{d}{dx}[x^3]\cdot\log_5 2x\\ &= x^3\cdot\frac{1}{x\ln 5} + 3x^2\cdot\log_5 2x\\ &= \frac{x^2}{\ln 5} + 3x^2\log_5 2x.\end{aligned}$$

Example 5:

Find the derivative of $y = \ln\frac{x}{x+1}$.

Solution:

We use the Quotient Rule and the natural logarithm rule:

$$\begin{aligned}y' &= \frac{1}{\frac{x}{x+1}}\cdot\frac{(x+1)(1)-(1)(x)}{(x+1)^2}\\ &= \frac{x+1}{x}\cdot\frac{1}{(x+1)^2}\\ &= \frac{1}{x(x+1)}.\end{aligned}$$

Integrals Involving Natural Logarithmic Function

In the last section, we have learned that the derivative of $y = \ln u(x)$ is $y' = \frac{1}{u(x)}.u'(x)$. The antiderivative is

$$\int\frac{u'(x)}{u(x)}dx = \ln|u(x)| + C.$$

If the argument of the natural logarithm is x, then $\frac{d}{dx}[\ln x] = 1/x$, thus

$$\int \frac{1}{x}dx = \ln|x|+C.$$

Example 6:

Evaluate $\int \frac{1}{x+1}dx$.

Solution:

In general, whenever you encounter an integral with an integrand as a rational function, it might be possible that it can be integrated with the rule of natural logarithm. To do so, determine the derivative of the denominator. If it is the numerator itself, then the integration is simply the ln of the absolute value of the denominator. Let's test this technique.

$$\int \frac{1}{x+1}dx.$$

Notice that the derivative of the denominator is 1, which is equal to the numerator. Thus the solution is simply the natural logarithm of the absolute value of the denominator:

$$\int \frac{1}{x+1}dx = \ln|x+1|+C.$$

The formal way of solving such integrals is to use $u-$substitution by letting u equal the denominator. Here, let $u = x+1$, and $du = dx$. Substituting,

$$\begin{aligned}\int \frac{1}{x+1}dx &= \int \frac{1}{u}du \\ &= \ln|u|+C \\ &= \ln|x+1|+C.\end{aligned}$$

Remark: The integral must use the absolute value symbol because although x may have negative values, the domain of $\ln(x)$ is restricted to $x \geq 0$.

Example 7:

Evaluate $\int \frac{4x+1}{4x^2+2x+1}dx$.

Solution:

As you can see here, the derivative of the denominator is $8x+2$. Our numerator is $4x+1$. However, when we multiply the numerator by 2, we get the derivative of the denominator. Hence

$$\begin{aligned}\int \frac{4x+1}{4x^2+2x+1}dx &= \frac{1}{2}\int \frac{2(4x+1)}{4x^2+2x+1}dx \\ &= \frac{1}{2}\int \frac{8x+2}{4x^2+2x+1}dx \\ &= \frac{1}{2}\ln|4x^2+2x+1|+C.\end{aligned}$$

Again, we could have used $u-$substitution.

Example 8:

Evaluate $\int \tan x dx$.

Solution:

To solve, we rewrite the integrand as

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Looking at the denominator, its derivative is $-\sin x$. So we need to insert a minus sign in the numerator:

$$\begin{aligned} &= -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln|\cos x| + C. \end{aligned}$$

Derivatives of Exponential Functions

We have discussed above that the exponential function is simply the inverse function of the logarithmic function. To obtain a derivative formula for the exponential function with base b, we rewrite $y = b^x$ as

$$x = \log_b y.$$

Differentiating implicitly,

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$ and replacing y with b^x,

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

Thus the derivative of an exponential function is

$$\frac{d}{dx}[b^x] = b^x \ln b.$$

In the special case where the base is $b^x = e^x$, since $\ln e = 1$ the derivative rule becomes

$$\frac{d}{dx}[e^x] = e^x.$$

To generalize, if u is a differentiable function of x, with the use of the Chain Rule the above derivatives take the general form

$$\frac{d}{dx}[b^u] = b^u \cdot \ln b \cdot \frac{du}{dx}.$$

And if $b = e$,

$$\frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx}.$$

Derivatives of Exponential Functions

$$\begin{aligned} \frac{d}{dx}[b^u] &= b^u \cdot \ln b \cdot \frac{du}{dx} = u' b^u \ln b \\ \frac{d}{dx}[e^u] &= e^u \cdot \frac{du}{dx} = u' e^u \end{aligned}$$

Example 9:

Find the derivative of $y = 2^{x^2}$.

Solution:

Applying the rule for differentiating an exponential function,

$$\begin{aligned} y' &= (2x)2^{x^2} \ln 2 \\ &= 2^{x^2+1} \cdot x \cdot \ln 2. \end{aligned}$$

Example 10:

Find the derivative of $y = e^{x^2}$.

Solution:

Since

$$\begin{aligned} \frac{d}{dx}[e^u] &= u' e^u, \\ y' &= 2xe^{x^2}. \end{aligned}$$

Example 11:

Find $f'(x)$ if

$$f(x) = \frac{1}{\sqrt{\pi\sigma}} e^{-\alpha k (x-x_0)^2}.$$

where σ, α, x_0, and k are constants and $\sigma \neq 0$.

Solution:

We apply the exponential derivative and the Chain Rule:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{\pi\sigma}}(-2\alpha k(x-x_0))e^{-\alpha k(x-x_0)^2} \\ &= -\frac{2\alpha k(x-x_0)}{\sqrt{\pi\sigma}}e^{-\alpha k(x-x_0)^2}. \end{aligned}$$

Integrals Involving Exponential Functions

Associated with the exponential derivatives in the box above are the two corresponding integration formulas:

$$\begin{aligned} \int b^u du &= \frac{1}{\ln b}b^u + C, \\ \int e^u du &= e^u + C. \end{aligned}$$

The following examples illustrate how they can be used.

Example 12:

Evaluate $\int 5^x dx$.

Solution:

$$\begin{aligned} \int 5^x dx &= \frac{1}{\ln 5}5^x + C \\ &= \frac{5^x}{\ln 5} + C. \end{aligned}$$

Example 13:

$$\int e^x dx.$$

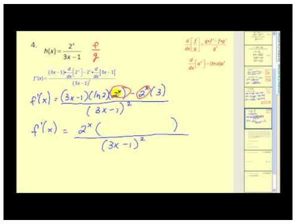
Solution:

$$\int e^x dx = e^x + C.$$

In the next chapter, we will learn how to integrate more complicated integrals, such as $\int x^2 e^{x^3} dx$, with the use of $u-$substitution and integration by parts along with the logarithmic and exponential integration formulas.

Multimedia Links

For a video presentation of the derivatives of exponential and logarithmic functions **(4.4)**, see Math Video Tutorials by James Sousa, The Derivatives of Exponential and Logarithmic Functions (8:26).



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Review Questions

1. Find dy/dx of $y = e^{6x}$
2. Find dy/dx of $y = e^{3x^3-2x^2+6}$
3. Find dy/dx of $y = e^{x^2} \cdot \ln\left(\frac{1}{x}\right)$
4. Find dy/dx of $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
5. Find dy/dx of $y = \cos(e^x)$
6. Find dy/dx of $y = \ln(\sin(\ln x))$
7. Evaluate $\int \frac{1}{e^x} dx$
8. Evaluate $\int \sqrt{e^x} dx$
9. Evaluate $\int \frac{4x-3}{4x^2-6x+7} dx$
10. Evaluate $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$
11. Evaluate $\int_0^e \frac{dx}{x+e}$
12. Evaluate $\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4} dx$

6.4 Exponential Growth and Decay

Learning Objectives

A student will be able to:

- Apply the laws of exponential and logarithmic functions to a variety of applications.
- Model situations of growth and decay in a variety of problems.

When the rate of change in a substance or population is proportional to the amount present at any time t, we say that this substance or population is going through either a decay or a growth, depending on the sign of the constant of proportionality.

This kind of growth is called ***exponential growth*** and is characterized by rapid growth or decay. For example, a population of bacteria may increase exponentially with time because the rate of change of its population is proportional to its population at a given instant of time (more bacteria make more bacteria and fewer bacteria make fewer bacteria). The decomposition of a radioactive substance is another example in which the rate of decay is proportional to the amount of the substance at a given time instant. In the business world, the interest added to an investment each day, month, or year is proportional to the amount present, so this is also an example of exponential growth.

Mathematically, the relationship between amount y and time t is a differential equation:

$$\frac{dy}{dt} = ky.$$

Separating variables,

$$\frac{dy}{y} = kdt,$$

and integrating both sides,

$$\int \frac{dy}{y} = \int kdt,$$

gives us

$$\begin{aligned} \ln y &= kt + C, \\ y &= e^{kt} e^{C} \\ &= Ce^{kt}. \end{aligned}$$

So the solution to the equation $(dy/dt) = ky$ has the form $y = Ce^{kt}$. The box below summarizes the details of this function.

The Law of Exponential Growth and Decay

The function $y = Ce^{kt}$ is a model for exponential growth or decay, depending on the value of k.

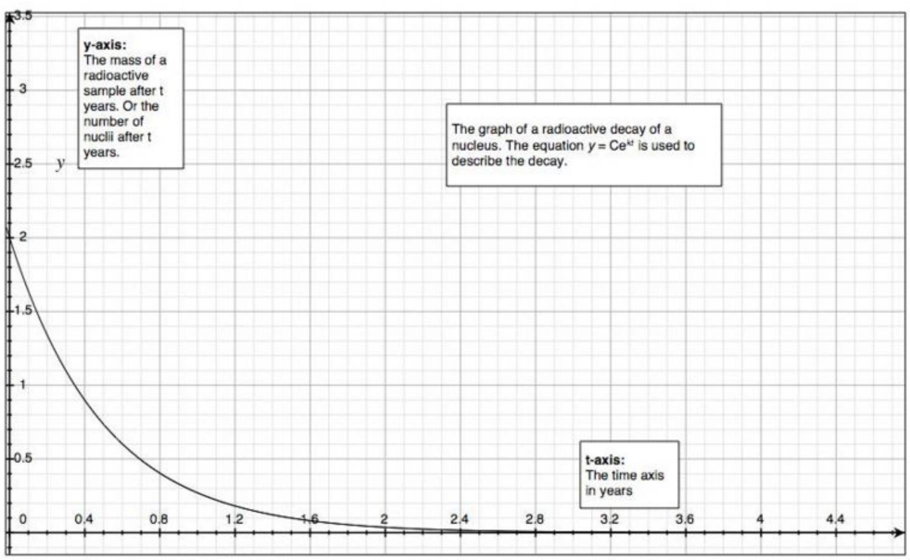
- If $k > 0$: The function represents exponential growth (increase).
- If $k < 0$: The function represents exponential decay (decrease).

Where t is the time, C is the initial population at $t = 0$, and y is the population after time t.

Applications of Growth and Decay

Radioactive Decay

In physics, radioactive decay is a process in which an unstable atomic nucleus loses energy by emitting radiation in the form of electromagnetic radiation (like gamma rays) or particles (such as beta and alpha particles). During this process, the nucleus will continue to decay, in a chain of decays, until a new stable nucleus is reached (called an isotope). Physicists measure the rate of decay by the time it takes a sample to lose half of its nuclei due to radioactive decay. Initially, as the nuclei begins to decay, the rate starts very fast and furious, but it slows down over time as more and more of the available nuclei have decayed. The figure below shows a typical radioactive decay of a nucleus. As you can see, the graph has the shape of an exponential function with $k < 0$.



The equation that is used for radioactive decay is $y = Ce^{kt}$. We want to find an expression for the half-life of an isotope. Since half-life is defined as the time it takes for a sample to lose half of its nuclei, then if we starting with an initial mass C (measured in grams), then after some time t,y will become half the amount that we started with, $C/2$. Substituting this into the exponential decay model,

$$y = Ce^{kt}$$

$$\frac{C}{2} = Ce^{kt}.$$

Canceling C from both sides,

$$\frac{1}{2} = e^{kt}.$$

Solving for t, which is the half-life, by taking the natural logarithm on both sides,

$$\begin{aligned}\ln\frac{1}{2} &= \ln e^{kt} \\ -\ln 2 &= kt.\end{aligned}$$

Solving for t, and denoting it with new notation $t_{1/2}$ for half-life (a standard notation in physics),

$$t_{1/2} = \frac{-\ln 2}{k} = \frac{-0.693}{k}$$

This is a famous expression in physics for measuring the half-life of a substance if the decay constant k is known. It can also be used to compute k if the half-life $t_{1/2}$ is known.

Example 1:

A radioactive sample contains 2 grams of nobelium. If you know that the half-life of nobelium is 25 seconds, how much will remain after 3 minutes?

Solution:

Before we compute the mass of nobelium after 3 minutes, we need to first know its decay rate k. Using the half-life formula,

$$\begin{aligned}t_{1/2} &= \frac{-\ln 2}{k} \\ k &= \frac{-\ln 2}{t_{1/2}} \\ &= \frac{-\ln 2}{25} \\ &= -0.028 \text{ sec}^{-1}\end{aligned}$$

So the decay rate is $k = -0.028/\text{sec}$. The common unit for the decay rate is the Becquerel (Bq): 1 Bq is equivalent to 1 decay per sec. Since we found k, we are now ready to calculate the mass after 3 minutes. We use the radioactive decay formula. Remember, C represents the initial mass, $C = 2$ grams, and $t = 3$ minutes $= 180$ seconds. Thus

$$\begin{aligned}y &= Ce^{kt} \\ &= 2e^{(-0.028)(180)} \\ &= 0.013 \text{ grams.}\end{aligned}$$

So after 3 minutes, the mass of the isotope is approximately 0.013 grams.

Population Growth

The same formula $y = Ce^{kt}$ can be used for population growth, except that $k > 0$, since it is an increasing function.

Example 2:

A certain population of bacteria increases continuously at a rate that is proportional to its present number. The initial population of the bacterial culture is 140 and jumped to 720 bacteria in 4 hours.

1. How many will be there in 10 hours?
2. How long will it take the population to double?

Solution:

From reading the first sentence in the problem, we learn that the bacteria is increasing exponentially. Therefore, the exponential growth formula is the correct model to use.

1. Just like we did in the previous example, we need to first find k, the growth rate. Notice that $C = 140, t = 4$, and $y = 720$. Substituting and solving for k.

$$\begin{aligned} y &= Ce^{kt} \\ 720 &= 140e^{k(4)}. \end{aligned}$$

Dividing both sides by 140 and then projecting the natural logarithm on both sides,

$$\begin{aligned} \ln \frac{720}{140} &= \ln e^{4k} \\ \ln 5.143 &= 4k \\ k &= 0.409. \end{aligned}$$

Now that we have found k, we want to know how many will be there after 10 hours. Substituting,

$$\begin{aligned} y &= Ce^{kt} \\ &= 140e^{(0.409)(10)} \\ &= 8364 \text{ bacteria}. \end{aligned}$$

2. We are looking for the time required for the population to double. This means that we are looking for the time at which $y = 2C$. Substituting,

$$\begin{aligned} y &= Ce^{kt} \\ 2C &= Ce^{kt} \\ 2 &= e^{kt}. \end{aligned}$$

Solving for t requires taking the natural logarithm of both sides:

$$\begin{aligned} \ln 2 &= \ln e^{kt} \\ \ln 2 &= kt. \end{aligned}$$

Solving for t,

$$\begin{aligned} t &= \frac{\ln 2}{k} \\ &= \frac{\ln 2}{0.409} \\ t &= 1.7 \text{ hours.} \end{aligned}$$

This tells us that after about 1.7 hours (around 100 minutes) the population of the bacteria will double in number.

Compound Interest

Investors and bankers depend on compound interest to increase their investment. Traditionally, banks added interest after certain periods of time, such as a month or a year, and the phrase was "the interest is being compounded monthly or yearly." With the advent of computers, the compunding could be done daily or even more often. Our exponential model represents continuous, or instantaneous, compounding, and it is a good model of current banking practices. Our model states that

$$A = Pe^{rt},$$

where P is the initial investment (present value) and A is the future value of the investment after time t at an interest rate of r. The interest rate r is usually given in percentage per year. The rate must be converted to a decimal number, and t must be expressed in years. The example below illustrates this model.

Example 3:

An investor invests an amount of $10,000 and discovers that its value has doubled in 5 years. What is the annual interest rate that this investment is earning?

Solution:

We use the exponential growth model for continuously compounded interest,

$$\begin{aligned} A &= Pe^{rt} \\ 20,000 &= 10,000e^{r(5)} \\ 2 &= e^{5r} \\ \ln 2 &= 5r. \end{aligned}$$

Thus

$$\begin{aligned} r &= \frac{\ln 2}{5} \\ &= 0.139 \\ r &= 13.9\% \end{aligned}$$

The investment has grown at a rate of 13.9% per year.

Example 4:

Going back to the previous example, how long will it take the invested money to triple?

Solution:

$$\begin{aligned}
A &= Pe^{rt} \\
30,000 &= 10,000e^{(0.139)(t)} \\
3 &= e^{0.139t} \\
\ln 3 &= 0.139t \\
t &= \frac{\ln 3}{0.139} \\
&= 7.9 \text{ years.}
\end{aligned}$$

Other Exponential Models and Examples

Not all exponential growths and decays are modeled in the natural base e or by $y = Ce^{kt}$. Actually, in everyday life most are constructed from empirical data and regression techniques. For example, in the business world the demand function for a product may be described by the formula

$$p = 12,400 - \frac{11,000}{2.2 + e^{-0.0003x}},$$

where p is the price per unit and x is the number of units produced. So if the business is interested in basing the price of its unit on the number that it is projecting to sell, this formula becomes very helpful.

If a motorcycle factory is projecting to sell 7000 units in one month, what price should the factory set on each motorcycle?

$$\begin{aligned}
p &= 12,400 - \frac{11,000}{2.2 + e^{0.0003x}} \\
&= 12,400 - \frac{11,000}{2.2 + e^{0.0003(7000)}} \\
&= 12,400 - \frac{11,000}{2.2 + 0.122} \\
&= 7,663.
\end{aligned}$$

Thus the factory's base price for each motorcycle should be set at $7663.

As another example, let's say a medical researcher is studying the spread of the flu virus through a certain campus during the winter months. Let's assume that the model for the spread is described by

$$P = \frac{4500}{1 + 4499e^{-0.8x}}, \; x \geq 0,$$

where P represents the total number of infected students and x is the time, measured in days. Suppose the researcher is interested in the number of students who will be infected in the next week (7 days). Substituting $x = 7$ into the model,

$$\begin{aligned}
P &= \frac{4500}{1+4499e^{-0.8x}} \\
&= \frac{4500}{1+4499e^{-0.8(7)}} \\
&= \frac{4500}{1+4499(0.004)} \\
&= 255.
\end{aligned}$$

According to the model, 255 students will become infected with the flu virus. Assume further that the researcher wants to know how long it will take until 1000 students become infected with the flu virus. Solving for x,

$$P = \frac{4500}{1+4499e^{-0.8x}}.$$

Cross-multiplying,

$$\begin{aligned}
P(1+4499e^{-0.8x}) &= 4500 \\
1+4499e^{-0.8x} &= \frac{4500}{P} \\
4499e^{-0.8x} &= \frac{4500}{P} - 1 \\
&= \frac{4500-P}{P} \\
e^{-0.8x} &= \frac{4500-P}{4499P}.
\end{aligned}$$

Projecting ln on both sides,

$$\begin{aligned}
-0.8x &= \ln\left[\frac{4500-P}{4499P}\right] \\
x &= \ln\left[\frac{4500-P}{4499P}\right] \div (-0.8).
\end{aligned}$$

Substituting for $P = 1000$,

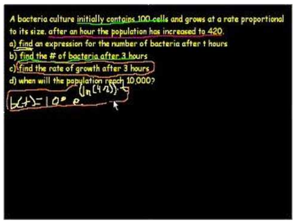
$$x = 9 \text{ days}.$$

So the flu virus will spread to 1000 students in 9 days.

Other applications are introduced in the exercises.

Multimedia Links

For a video presentation of exponential growth involving bacteria (some calculus in part c) **(14.0)**, see Khan Academy, Exponential Growth and Decay (16:00).



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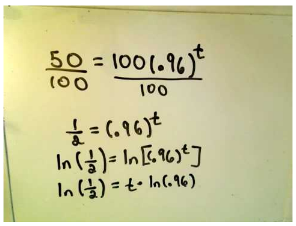
For a video presentation of exponential decay **(14.0)**, see Just Math Tutoring, Exponential Decay, Finding Half-Life (6:08).



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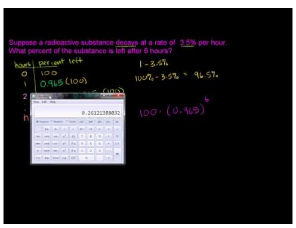
For additional problems on exponential growth and decay **(14.0)**, see Khan Academy, Word Problem Solving, Exponential Growth and Decay (7:21).



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Review Questions

1. In 1990, the population of the USA was 249 million. Assume that the annual growth rate is 1.8%.
   a. According to this model, what was the population in the year 2000?
   b. According to this model, in which year will the population reach 1 billion?
2. Prove that if a quantity A is exponentially growing and if A_1 is the value at t_1 and A_2 at time t_2, then the growth rate will be given by

$$k = \frac{1}{t_1 - t_2} \ln\left(\frac{A_1}{A_2}\right).$$

3. Newton's Law of Cooling states that the rate of cooling is proportional to the difference in temperature between the object and the surroundings. The law is expressed by the formula $T(t) = (T_0 - T_r)e^{-kt} + T_r$ where T_0 is the initial temperature of the object at $t = 0, T_r$ is the room temperature (the surroundings), and k is a constant that is unique for the measuring instrument (the thermometer) called the *time constant.* Suppose a liter of juice at $23°C$ is placed in the refrigerator to cool. If the temperature of the refrigerator is kept at $11°C$ and $k = 0.417$, what is the temperature of the juice after 3 minutes?
4. Referring back to problem 3, if it takes an object 320 seconds to cool from $40°C$ above room temperature to $22°C$ above room temperature, how long will it take to cool another $10°C$ after it reaches $22°C$ above room temperature?

5. Polonium−210 is a radioactive isotope with half-life of 140 days. If a sample has a mass of 10 grams, how much will remain after 10 weeks?
6. In the physics of acoustics, there is a relationship between the subjective sensation of loudness and the physically measured intensity of sound. This relationship is called the *sound level* β. It is specified on a logarithmic scale and measured with units of **decibels** (dB). The sound level β of any sound is defined in terms of its intensity I (in the SI-mks unit system, it is measured in watts per meter squared, W/m^2) as

$$\beta = 10\log\frac{I}{10^{-12}}.$$

For example, the average decibel level of a busy street traffic is 70 dB, normal conversation at a dinner table is 55 dB, the sound of leaves rustling is 10 dB, the siren of a fire truck at 30 meters is 100 dB, and a loud rock concert is 120 dB. The sound level 120 dB is considered the threshold of pain for the human ear and 0 dB is the threshold of hearing (the minimum sound that can be heard by humans.)

   a. If at a heavy metal rock concert a dB meter registered 130 dB, what is the intensity I of this sound level?
   b. What is the sound level (in dB) of a sound whose intensity is $2.0 \times 10^{-6}\ \text{W}/\text{m}^2$?

7. Referring to problem #6, a single mosquito 10 meters away from a person makes a sound that is barely heard by the person (threshold 0 dB). What will be the sound level of 1000 mosquitoes at the same distance?
8. Referring back to problem #6, a noisy machine at a factory produces a sound level of 90 dB. If an identical machine is placed beside it, what is the combined sound level of the two machines?

6.5 Derivatives and Integrals Involving Inverse Trigonometric Functions

Learning Objectives

A student will be able to:

- Learn the basic properties inverse trigonometric functions.
- Learn how to use the derivative formula to use them to find derivatives of inverse trigonometric functions.
- Learn to solve certain integrals involving inverse trigonometric functions.

A Quick Algebraic Review of Inverse Trigonometric Functions

You already know what a trigonometric function is, but what is an inverse trigonometric function? If we ask what is $\sin(\pi/6)$ equal to, the answer is $(1/2)$. That is simple enough. But what if we ask what angle has a sine of $(1/2)$? That is an inverse trigonometric function. So we say $\sin(\pi/6) = (1/2)$, but $\sin^{-1}(1/2) = (\pi/6)$. The "$\sin^{-1}$" is the notation for the inverse of the sine function. For every one of the six trigonometric functions there is an associated inverse function. They are denoted by

$$\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x, \csc^{-1} x, \cot^{-1} x$$

Alternatively, you may see the following notations for the above inverses, respectively,

$$\arcsin x, \arccos x, \arctan x, \operatorname{arcsec} x, \operatorname{arccsc} x, \operatorname{arccot} x$$

Since all trigonometric functions are periodic functions, they do not pass the horizontal line test. Therefore they are not one-to-one functions. The table below provides a brief summary of their definitions and basic properties. We will restrict our study to the first four functions; the remaining two, $\csc^{-1}$ and $\cot^{-1}$, are of lesser importance (in most applications) and will be left for the exercises.

TABLE 6.1:

Inverse Function	Domain	Range	Basic Properties
$\sin^{-1}$	$-1 \le x \le 1$	$\frac{-\pi}{2} \le y \le \frac{\pi}{2}$	$\sin(\sin^{-1}(x)) = x$
$\cos^{-1}$	$-1 \le x \le 1$	$0 \le y \le \pi$	$\cos^{-1}(\cos x) = \cos(\cos^{-1}(x)) = x$
$\tan^{-1}$	all R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$	$\tan^{-1}(\tan x) = \tan(\tan^{-1}(x)) = x$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$\sec^{-1}(\sec x) = \sec(\sec^{-1}(x)) = x$

The range is based on limiting the domain of the original function so that it is a one-to-one function.

Example 1:

What is the exact value of $\sin^{-1}(\sqrt{3}/2)$?

Solution:

This is equivalent to $\sin x = \frac{\sqrt{3}}{2}$. Thus $\sin^{-1}(\sqrt{3}/2) = \pi/3$. You can easily confirm this result by using your scientific calculator.

Example 2:

Most calculators do not provide a way to calculate the inverse of the secant function, $\sec^{-1} x$. A practical trick however is to use the identity

$$\sec^{-1} x = \cos^{-1} \frac{1}{x}$$

(Recall that $\sec\theta = \frac{1}{\cos\theta}$.)

For practice, use your calculator to find $\sec^{-1}(3.24)$.

Solution:

Since

$$\frac{1}{x} = \frac{1}{3.24} = 0.3086,$$

$$\sec^{-1} 3.24 = \cos^{-1} 0.3086 = 72^\circ.$$

Here are two other identities that you may need to enter into your calculator:

$$\csc^{-1} x = \sin^{-1} \frac{1}{x},$$
$$\cot^{-1} x = \tan^{-1} \frac{1}{x}.$$

The Derivative Formulas of the Inverse Trigonometric Functions

If u is a differentiable function of x then the generalized derivative formulas for the inverse trigonometric functions are (we introduce them here without a proof):

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} -1 < u < 1$$
$$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} -1 < u < 1$$
$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\frac{du}{dx} -\infty < x < \infty$$
$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} |u| > 1$$
$$\frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} |u| > 1$$
$$\frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{1+u^2}\frac{du}{dx} -\infty < x < \infty$$

Example 3:

Differentiate $y = \sin^{-1}(2x^4)$

Solution:

Let $u = 2x^4$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-(2x^4)^2}} \cdot (8x^3) \\ &= \frac{8x^3}{\sqrt{1-4x^8}}.\end{aligned}$$

Example 4:

Differentiate $\tan^{-1}(e^{3x})$.

Solution:

Let $u = e^{3x}$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+(e^{3x})^2} \cdot 3e^{3x} \\ &= \frac{3e^{3x}}{1+e^{6x}}.\end{aligned}$$

Example 5:

Find dy/dx if $y = \cos^{-1}(\sin x)$.

Solution:

Let $u = \sin x$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{-1}{\sqrt{1-\sin^2 x}} \\ &= \frac{-1}{\cos x}.\end{aligned}$$

The Integration Formulas of the Inverse Trigonometric Functions

The derivative formulas in the box above yield the following integrations formulas for inverse trigonometric functions:

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$$
$$\int \frac{du}{1+u^2} = \tan^{-1} u + c$$
$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + c$$

Example 6:

Evaluate $\int \frac{dx}{1+4x^2}$.

Solution:

Before we integrate, we use $u-$substitution. Let $u = 2x$ (the square root of $4x^2$). Then $du = 2dx$. Substituting,

$$\begin{aligned}\int \frac{dx}{1+4x^2} &= \int \frac{1/2}{1+u^2} du \\ &= \frac{1}{2}\int \frac{1}{1+u^2} du \\ &= \frac{1}{2}\tan^{-1} u + c \\ &= \frac{1}{2}\tan^{-1}(2x) + c.\end{aligned}$$

Example 7:

Evaluate $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx$.

Solution:

We use $u-$substitution. Let $u = e^x$, so $du = e^x dx$. Substituting,

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}} dx &= \int \frac{e^x}{\sqrt{1-u^2}} \frac{du}{e^x} \\ &= \int \frac{1}{\sqrt{1-u^2}} du \\ &= \sin^{-1} u + c \\ &= \sin^{-1}(e^x) + c.\end{aligned}$$

Example 8:

Evaluate the definite integral $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx$.

Solution:

Substituting $u = e^{-x}, du = -e^{-x} dx$.

To change the limits,

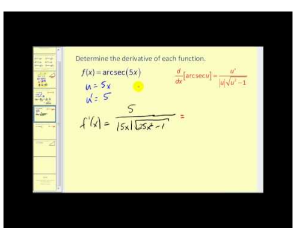
$$x = \ln 2 \to u = e^{-x} = e^{-\ln 2} = e^{\ln 1/2} = \frac{1}{2},$$
$$x = \ln\left(\frac{2}{\sqrt{3}}\right) \to u = e^{-x} = e^{-\ln 2/\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

Thus our integral becomes

$$\begin{aligned}
\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx &= \int_{1/2}^{\sqrt{3}/2} \frac{u}{\sqrt{1-u^2}} \frac{-du}{u} \\
&= -\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}} du \\
&= -\left[\sin^{-1} u\right]_{1/2}^{\sqrt{3}/2} \\
&= -\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right] \\
&= -\left[\frac{\pi}{3} - \frac{\pi}{6}\right] \\
&= -\frac{\pi}{6}.
\end{aligned}$$

Multimedia Links

For a video presentation of the derivatives of inverse trigonometric functions **(4.4)**, see Math Video Tutorials by James Sousa, The Derivatives of Inverse Trigonometric Functions (8:55).



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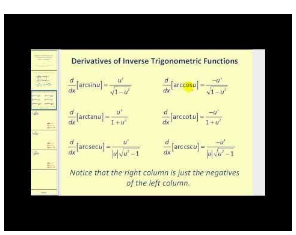
For three presentations of integration involving inverse trigonometric functions **(18.0)**, see Math Video Tutorials by James Sousa, Integration Involving Inverse Trigonometric Functions, Part1 (7:39)



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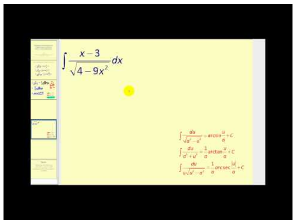
; Math Video Tutorials by James Sousa, Integration Involving Inverse Trigonometric Functions, Part2 (6:39)



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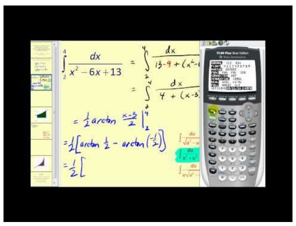
; This last video includes an example showing completing the square **(19.0)**, Math Video Tutorials by James Sousa, Integration Involving Inverse Trigonometric Functions, Part3 (6:18).



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Review Questions

1. Find dy/dx of $y = \sec^{-1} x^2$.
2. Find dy/dx of $y = \frac{1}{\tan^{-1} x}$.
3. Find dy/dx of $y = \ln(\cos^{-1} x)$.
4. Find dy/dx of $y = \sin^{-1} e^{-4x}$.
5. Find dy/dx of $y = \sin^{-1}(x^2 \ln x)$.
6. Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.
7. Evaluate $\int_1^3 \frac{dx}{\sqrt{x}(x+1)}$.
8. Evaluate $\int \frac{x-3}{x^2+1} dx$.
9. Evaluate $\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2} dx$.
10. Given the points $A(2,1)$ and $B(5,4)$, find a point Q in the interval $[2,5]$ on the $x-$axis that maximizes angle $\measuredangle AQB$.

6.6 L'Hospital's Rule

Learning Objectives

A student will be able to:

- Learn how to find the limit of indeterminate form $(0/0)$ by L'Hospital's rule.

If the two functions $f(x)$ and $g(x)$ are both equal to zero at $x = a$, then the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by directly substituting $x = a$. The reason is because when we substitute $x = a$, the substitution will produce $(0/0)$, known as an ***indeterminate form***, which is a meaningless expression. To work around this problem, we use L'Hospital's rule, which enables us to evaluate limits of indeterminate forms.

L'Hospital's Rule

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, and $f'(a)$ and $g'(a)$ exist, where $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The essence of L'Hospital's rule is to be able to replace one limit problem with a simpler one. In each of the examples below, we will employ the following three-step process:

1. Check that $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form $0/0$. To do so, directly substitute $x = a$ into $f(x)$ and $g(x)$. If you get $f(a) = g(a) = 0$, then you can use L'Hospital's rule. Otherwise, it cannot be used.
2. Differentiate $f(x)$ and $g(x)$ separately.
3. Find $\lim_{x\to a} \frac{f'(x)}{g'(x)}$. If the limit is finite, then it is equal to the original limit $\lim_{x\to a} \frac{f(x)}{g(x)}$.

Example 1:

Find $\lim_{x\to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

Solution:

When $x = 0$ is substituted, you will get $0/0$.

Therefore L'Hospital's rule applies:

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} &= \lim_{x\to 0}\left[\frac{\frac{d}{dx}(\sqrt{2+x}-\sqrt{2})}{\frac{d}{dx}(x)}\right]\\ &= \left[\frac{1/(2\sqrt{2+x})}{1}\right]_{x=0}\\ &= \frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{2}}{4}.\end{aligned}$$

Example 2:

Find $\lim_{x\to 0}\frac{1-\cos 2x}{x^2+2x}$

Solution:

We can see that the limit is $0/0$ when $x=0$ is substituted.

Using L'Hospital's rule,

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos 2x}{x^2+2x} &= \left[\frac{2\sin 2x}{2x+2}\right]_{x=0}\\ &= 0/2\\ &= 0.\end{aligned}$$

Example 3:

Use L'Hospital's rule to evaluate $\lim_{x\to 3}\frac{x^2-9}{x-3}$.

Solution:

$$\lim_{x\to 3} = \frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{2x}{1} = 6.$$

Example 4:

Evaluate $\lim_{x\to 0}\frac{\sin 3x}{x}$.

Solution:

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\cos 3x}{1} = 3.$$

Example 5:

Evaluate $\lim_{x\to\pi/2}\frac{5-5\sin x}{\cos x}$.

Solution:

We can use L'Hospital's rule since the limit produces the $0/0$ once $x=\pi/2$ is substituted. Hence

$$\lim_{x\to\pi/2}\frac{5-5\sin x}{\cos x} = \lim_{x\to\pi/2}\frac{0-5\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

A broader application of L'Hospital's rule is when $x = a$ is substituted into the derivatives of the numerator and the denominator but both still equal zero. In this case, a second differentiation is necessary.

Example 6:

Evaluate $\lim_{x\to 0} \frac{e^x - \cos x}{x^2}$.

Solution:

$$\lim_{x\to 0} \frac{e^x - \cos x}{x^2} = \lim_{x\to 0} \frac{e^x + \sin x}{2x}.$$

As you can see, if we apply the limit at this stage the limit is still indeterminate. So we apply L'Hospital's rule again:

$$\begin{aligned} &= \lim_{x\to 0} \frac{e^x - \cos x}{2} \\ &= \frac{1-1}{2} = \frac{0}{2} = 0. \end{aligned}$$

Review Questions

Find the limits.

1. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$
2. $\lim_{x\to 1} \frac{\ln x}{\tan \pi x}$
3. $\lim_{x\to 0} \frac{e^{10x} - e^{6x}}{x}$
4. $\lim_{x\to \pi} \frac{\sin x}{x - \pi}$
5. $\lim_{x\to 0} \frac{xe^x}{1 - e^x}$
6. If k is a nonzero constant and $x > 0$.
   a. Show that $\int_1^x \frac{1}{t^{1-k}} dt = \frac{x^k - 1}{k}$.
   b. Use L'Hospital's rule to find $\lim_{k\to 0} \frac{x^k - 1}{k}$.
7. *Cauchy's Mean Value Theorem* states that if the functions f and g are continuous on the interval (a, b) and $g' \neq 0$, then there exists a number c such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

   Find all possible values of c in the interval (a, b) that satisfy this property for

$$\begin{aligned} f(x) &= \cos x \\ g(x) &= \sin x \end{aligned}$$

   on the interval

$$[a, b] = \left[0, \frac{\pi}{2}\right].$$

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9731 .

CHAPTER 7 Integration Techniques

Chapter Outline

7.1 Integration by Substitution

Each basic rule of integration that you have studied so far was derived from a corresponding differentiation rule. Even though you have learned all the necessary tools for differentiating exponential, logarithmic, trigonometric, and algebraic functions, your set of tools for integrating these functions is not yet complete. In this chapter we will explore different ways of integrating functions and develop several integration techniques that will greatly expand the set of integrals to which the basic integration formulas can be applied. Before we do that, let us review the basic integration formulas that you are already familiar with from previous chapters.

1. The Power Rule $(n \neq -1)$:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

2. The General Power Rule $(n \neq -1)$:

$$\int u^n \frac{du}{dx} dx = \int u^n du = \frac{u^{n+1}}{n+1} + C.$$

3. The Simple Exponential Rule:

$$\int e^x dx = e^x + C.$$

4. The General Exponential Rule:

$$\int e^u \frac{du}{dx} dx = \int e^u du = e^u + C.$$

5. The Simple Log Rule:

$$\int \frac{1}{x} dx = \ln|x| + C.$$

6. The General Log Rule:

$$\int \frac{du/dx}{u} dx = \int \frac{1}{u} du = \ln|u| + C.$$

It is important that you remember the above rules because we will be using them extensively to solve more complicated integration problems. The skill that you need to develop is to determine which of these basic rules is needed to solve an integration problem.

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the technique of $u-$substitution.
- Apply the $u-$substitution technique to definite integrals.
- Apply the $u-$substitution technique to trig functions.

Probably one of the most powerful techniques of integration is *integration by substitution.* In this technique, you choose part of the integrand to be equal to a variable we will call u and then write the entire integrand in terms of u. The difficulty of the technique is deciding which term in the integrand will be best for substitution by u. However, with practice, you will develop a skill for choosing the right term.

Recall from Chapter 2 that if u is a differentiable function of x and if n is a real number and $n \neq -1$, then the Chain Rule tells us that

$$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}.$$

The reverse of this formula is the integration formula,

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1.$$

Sometimes it is not easy to integrate directly. For example, look at this integral:

$$\int (5x-2)^2 dx.$$

One way to integrate is to first expand the integrand and then integrate term by term.

$$\begin{aligned}\int (5x-2)^2 dx &= \int (25x^2 - 20x + 4) dx \\ &= 25\int x^2 dx - 20\int x dx + \int 4dx \\ &= \frac{25}{3}x^3 - 10x^2 + 4x + C.\end{aligned}$$

That is easy enough. However, what if the integral was

$$\int (5x-2)^{15} dx?$$

Would you still expand the integrand and then integrate term by term? That would be impractical and time-consuming. A better way of doing this is to change the variables. Changing variables can often turn a difficult integral, such as the one above, into one that is easy to integrate. The method of doing this is called ***integration by substitution,*** or for short, the u ***-substitution method***. The examples below will show you how the method is used.

Example 1:

Evaluate $\int (x+1)^5 dx$.

Solution:

Let $u = x+1$. Then $du = d(x+1) = 1dx = dx$. Substituting for u and du we get

$$\int (x+1)^5 dx = \int u^5 du.$$

Integrating using the power rule,

$$= \frac{u^6}{6} + C.$$

Since $u = x+1$, substituting back,

$$= \frac{(x+1)^6}{6} + C.$$

Example 2:

Evaluate $\int \sqrt{4x+3} dx$.

Solution:

Let $u = 4x+3$. Then $du = 4dx$. Solving for dx,

$$dx = du/4.$$

Substituting,

$$\begin{aligned} &= \int u^{1/2} \cdot \frac{1}{4} dx \\ &= \frac{1}{4} \int u^{1/2} dx \\ &= \frac{1}{4} \frac{u^{3/2}}{3/2} + C. \end{aligned}$$

Simplifying,

$$\begin{aligned} &= \frac{1}{6} u^{3/2} + C \\ &= \frac{1}{6} (4x+3)^{3/2} + C. \end{aligned}$$

Trigonometric Integrands

We can apply the change of variable technique to trigonometric functions as long as u is a differentiable function of x. Before we show how, recall the basic trigonometric integrals:

$$\begin{aligned}
\int \cos u du &= \sin u + C, \\
\int \sin u du &= -\cos u + C, \\
\int \sec^2 u du &= \tan u + C, \\
\int \csc^2 u du &= -\cot u + C, \\
\int (\sec u)(\tan u) du &= \sec u + C, \\
\int (\csc u)(\cot u) du &= -\csc u + C.
\end{aligned}$$

Example 3:

Evaluate $\int \cos(3x+2)dx$.

Solution:

The argument of the cosine function is $3x+2$. So we let $u = 3x+2$. Then $du = 3dx$, or $dx = du/3$.

Substituting,

$$\begin{aligned}
\int \cos(3x+2)dx &= \int \cos u \cdot \frac{1}{3} dx \\
&= \frac{1}{3} \int \cos u dx.
\end{aligned}$$

Integrating,

$$\begin{aligned}
&= \frac{1}{3} \sin u + C \\
&= \frac{1}{3} \sin(3x+2) + C.
\end{aligned}$$

Example 4:

This example requires us to use trigonometric identities before we substitute. Evaluate

$$\int \frac{1}{cos^2 3x} dx.$$

Solution:

Since $\sec 3x = \frac{1}{\cos 3x}$, the integral becomes

$$\int \frac{1}{cos^2 3x} dx = \int \sec^2 3x dx.$$

Substituting for the argument of the secant, $u = 3x$, then $du = 3dx$, or $dx = du/3$. Thus our integral becomes,

$$\begin{aligned}\int \sec^2 u.\frac{1}{3} du &= \frac{1}{3}\int \sec^2 u du \\ &= \frac{1}{3}\tan u + C \\ &= \frac{1}{3}\tan(3x) + C.\end{aligned}$$

Some integrations of trigonometric functions involve the logarithmic functions as a solution, as shown in the following example.

Example 5:

Evaluate $\int \tan x dx$.

Solution:

As you may have guessed, this is not a straightforward integration. We need to make use of trigonometric identities to simplify it. Since $\tan x = \sin x / \cos x$,

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Now make a change of variable x. Choose $u = \cos x$. Then $du = -\sin x dx$, or $dx = -du/\sin x$. Substituting,

$$\begin{aligned}\int \frac{\sin x}{\cos x} dx &= \int \frac{\sin x}{u}\left(\frac{-du}{\sin x}\right) \\ &= -\int \frac{du}{u}.\end{aligned}$$

This integral should look obvious to you. The integrand is the derivative of the natural logarithm $\ln u$.

$$\begin{aligned}&= -\ln|u| + C \\ &= -\ln|\cos x| + C.\end{aligned}$$

Another way of writing it, since $-\ln|u| = \ln\frac{1}{|u|}$, is

$$\begin{aligned}&= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

Using Substitution on Definite Integrals

Example 6:

Evaluate $\int_1^3 \frac{x}{\sqrt{2x-1}}dx$.

Solution:

Let $u = 2x-1$. Then $du = 2dx$, or $dx = du/2$. Before we substitute, we need to determine the new limits of integration in terms of the u variable. To do so, we simply substitute the limits of integration into $u = 2x-1$:

Lower limit: For $x = 1, u = 2(1)-1 = 1$.

Upper limit: For $x = 3, u = 2(3)-1 = 5$.

We now substitute u and the associated limits into the integral:

$$\int_1^5 \frac{x}{\sqrt{u}}\frac{du}{2}.$$

As you may notice, the variable x is still hanging there. To write it in terms of u, since $u = 2x-1$, solving for x, we get, $x = (u+1)/(2)$. Substituting back into the integral,

$$\begin{aligned}
&= \int_1^5 \frac{u+1}{2\sqrt{u}}\frac{du}{2} \\
&= \frac{1}{4}\int_1^5 \frac{u+1}{\sqrt{u}}du \\
&= \frac{1}{4}\int_1^5 (u+1)u^{-1/2}du \\
&= \frac{1}{4}\int_1^5 (u^{1/2}+u^{-1/2})du \\
&= \frac{1}{4}\left[\frac{2u^{3/2}}{3}+\frac{2u^{1/2}}{1}\right]_1^5.
\end{aligned}$$

Applying the Fundamental Theorem of Calculus by inserting the limits of integration and calculating,

$$= \frac{1}{4}\left(\left[\frac{2(5)^{3/2}}{3}+\frac{2(5)^{1/2}}{1}\right]-\left[\frac{2(1)^{3/2}}{3}+\frac{2(1)^{1/2}}{1}\right]\right).$$

Calculating and simplifying, we get

$$= \frac{4\sqrt{5}-2}{3}.$$

We could have chosen $u = \sqrt{2x-1}$ instead. You may want to try to solve the integral with this substitution. It might be easier and less tedious.

Example 7:

Let's try the substitution method of definite integrals with a trigonometric integrand.

Evaluate $\int_0^{\pi/4} \tan x \sec^2 x dx$.

Solution:

Try $u = \tan x$. Then $du = \sec^2 x dx, dx = du/\sec^2 x$.

Lower limit: For $x = 0, u = \tan 0 = 0$.

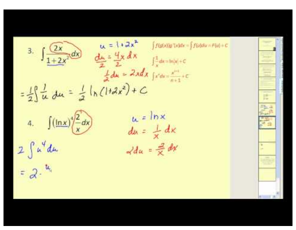
Upper limit: For $x = \pi/4, u = \tan \frac{\pi}{4} = 1$.

Thus

$$\begin{aligned}\int_0^{\pi/4} \tan x \sec^2 x dx &= \int_0^1 u du \\ &= \left[\frac{u^2}{2}\right]_0^1 \\ &= \frac{1}{2} - 0 = \frac{1}{2}.\end{aligned}$$

Multimedia Links

For video presentations on integration by substitution **(17.0)**, see Math Video Tutorials by James Sousa, Integration by Substitution, Part 1 of 2 (9:42)



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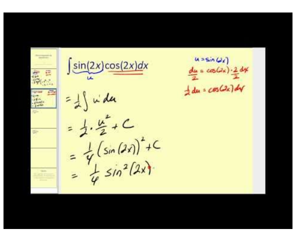
and Math Video Tutorials by James Sousa, Integration by Substitution, Part 2 of 2 (8:17).



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Review Questions

In the following exercises, evaluate the integrals.

1. $\int \frac{3}{(x-8)^2} dx$

2. $\int \sqrt{2+x}dx$
3. $\int \frac{1}{\sqrt{2+x}}dx$
4. $\int \frac{x^2}{x+1}dx$
5. $\int \frac{e^{-x}}{e^{-x}+2}dx$
6. $\int \frac{3\sqrt{t}+5}{t}dt$
7. $\int \frac{2}{\sqrt{3x-1}}dx$
8. $\int \sin x \cos x dx$
9. $\int \cos x \sqrt{1-\cos^2 x} dx$
10. $\int \sin^5 x \cos x dx$
11. $\int x^3 \cos 4x^4 dx$
12. $\int \sec^2(2x+4)dx$
13. $\int_0^2 xe^{x^2} dx$
14. $\int_0^{\sqrt{\pi}} x \sin x^2 dx$
15. $\int_0^1 x(x+5)^4 dx$

7.2 Integration By Parts

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Integration by Parts.
- Combine this technique with the $u-$substitution method to solve integrals.
- Learn to tabulate the technique when it is repeated.

In this section we will study a technique of integration that involves the product of algebraic and exponential or logarithmic functions, such as

$$\int x \ln x dx$$

and

$$\int xe^x dx.$$

Integration by parts is based on the product rule of differentiation that you have already studied:

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}.$$

If we integrate each side,

$$\begin{aligned} uv &= \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \\ &= \int udv + \int vdu. \end{aligned}$$

Solving for $\int udv$,

$$\int udv = uv - \int vdu.$$

This is the formula for integration by parts. With the proper choice of u and dv, the second integral may be easier to integrate. The following examples will show you how to properly choose u and dv.

Example 1:

Evaluate $\int x \sin x dx$.

Solution:

We use the formula $\int u dv = uv - \int v du$.

Choose

$$u = x$$

and

$$dv = \sin x dx.$$

To complete the formula, we take the differential of u and the simplest antiderivative of $dv = \sin x dx$.

$$\begin{aligned} du &= dx \\ v &= -\cos x. \end{aligned}$$

The formula becomes

$$\begin{aligned} \int x \sin x dx &= -x \cos x - \int (-\cos x) dx \\ &= -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

A Guide to Integration by Parts

Which choices of u and dv lead to a successful evaluation of the original integral? In general, choose u to be something that simplifies when differentiated, and dv to be something that remains manageable when integrated. Looking at the example that we have just done, we chose $u = x$ and $dv = \sin x dx$. That led to a successful evaluation of our integral. However, let's assume that we made the following choice,

$$\begin{aligned} u &= \sin x \\ dv &= x dx. \end{aligned}$$

Then

$$\begin{aligned} du &= \cos x dx \\ v &= x^2/2. \end{aligned}$$

Substituting back into the formula to integrate, we get

$$\begin{aligned} \int u dv &= uv - \int v du \\ &= \sin x \frac{x^2}{2} - \int \frac{x^2}{2} \cos x dx \end{aligned}$$

As you can see, this integral is worse than what we started with! This tells us that we have made the wrong choice and we must change (in this case switch) our choices of u and dv.

Remember, the goal of the integration by parts is to start with an integral in the form $\int udv$ that is hard to integrate directly and change it to an integral $\int vdu$ that looks easier to evaluate. However, here is a general guide that you may find helpful:

1. Choose dv to be the more complicated portion of the integrand that fits a basic integration formula. Choose u to be the remaining term in the integrand.
2. Choose u to be the portion of the integrand whose derivative is simpler than u. Choose dv to be the remaining term.

Example 2:

Evaluate $\int xe^x dx$.

Solution:

Again, we use the formula $\int udv = uv - \int vdu$.

Let us choose

$$u = x$$

and

$$dv = e^x dx.$$

We take the differential of u and the simplest antiderivative of $dv = e^x dx$:

$$\begin{aligned} du &= dx \\ v &= e^x. \end{aligned}$$

Substituting back into the formula,

$$\begin{aligned} \int udv &= uv - \int vdu \\ &= xe^x - \int e^x dx. \end{aligned}$$

We have made the right choice because, as you can see, the new integral $\int vdu = \int e^x dx$ is definitely simpler than our original integral. Integrating, we finally obtain our solution

$$\int xe^x dx = xe^x - e^x + C.$$

Example 3:

Evaluate $\int \ln x dx$.

Solution:

Here, we only have one term, $\ln x$. We can always assume that this term is multiplied by 1:

$$\int \ln x 1 dx.$$

So let $u = \ln x$, and $dv = 1dx$. Thus $du = 1/xdx$ and $v = x$. Substituting,

$$\begin{aligned}\int udv &= uv - \int vdu \\ \int \ln xdx &= x\ln x - \int x\frac{1}{x}dx \\ &= x\ln x - \int 1dx \\ &= x\ln x - x + C.\end{aligned}$$

Repeated Use of Integration by Parts

Oftentimes we use integration by parts more than once to evaluate the integral, as the example below shows.

Example 4:

Evaluate $\int x^2e^x dx$.

Solution:

With $u = x^2, dv = e^x dx, du = 2xdx$, and $v = e^x$, our integral becomes

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx.$$

As you can see, the integral has become less complicated than the original, $x^2e^x \to xe^x$. This tells us that we have made the right choice. However, to evaluate $\int xe^x dx$ we still need to integrate by parts with $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$, and

$$\begin{aligned}\int x^2e^x dx &= x^2e^x - 2\int xe^x dx \\ &= x^2e^x - 2\left[uv - \int udv\right] \\ &= x^2e^x - 2\left[xe^x - \int e^x dx\right] \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

Actually, the method that we have just used works for any integral that has the form $\int x^n e^x dx$, where n is a positive integer. The following section illustrates a systematic way of solving repeated integrations by parts.

Tabular Integration by Parts

Sometimes, we need to integrate by parts several times. This leads to cumbersome calculations. In situations like these it is best to organize our calculations to save us a great deal of tedious work and to avoid making unpredictable mistakes. The example below illustrates the method of ***tabular integration.***

Example 5:

Evaluate $\int x^2 \sin 3x dx$.

Solution:

Begin as usual by letting $u = x^2$ and $dv = \sin 3x dx$. Next, create a table that consists of three columns, as shown below:

TABLE 7.1:

Alternate signs	u and its derivatives	dv and its antiderivatives
$+$	$x^2 \searrow$	$\sin 3x$
$-$	$2x \searrow$	$\frac{-1}{3}\cos 3x$
$+$	$2 \searrow$	$\frac{-1}{9}\sin 3x$
$-$	0	$\frac{1}{27}\cos 3x$

To find the solution for the integral, pick the sign from the first row $(+)$, multiply it by u of the first row (x^2) and then multiply by the dv of the second row, $-1/3\cos 3x$ (watch the direction of the arrows.) This is the first term in the solution. Do the same thing to obtain the second term: Pick the sign from the second row, multiply it by the u of the same row and then follow the arrow to multiply the product by the dv in the third row. Eventually we obtain the solution

$$\int x^2 \sin 3x dx = \frac{-1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27}\cos 3x + C.$$

Solving for an Unknown Integral

There are some integrals that require us to evaluate two integrations by parts, followed by solving for the unknown integral. These kinds of integrals crop up often in electrical engineering and other disciplines.

Example 6:

Evaluate $\int e^x \cos x dx$.

Solution:

Let $u = e^x$, and $dv = \cos x dx$. Then $du = e^x dx, v = \sin x$, and

$$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx.$$

Notice that the second integral looks the same as our original integral in form, except that it has a $\sin x$ instead of $\cos x$. To evaluate it, we again apply integration by parts to the second term with $u = e^x, dv = \sin x dx, v = -\cos x$, and $du = e^x dx$.

Then

$$\int e^x \cos x dx = e^x \sin x - \left[-e^x \cos x - \int (-\cos x)(e^x dx)\right]$$
$$= e^x \sin x - e^x \cos x - \int e^x \cos x dx.$$

Notice that the unknown integral now appears on both sides of the equation. We can simply move the unknown integral on the right to the left side of the equation, thus adding it to our original integral:

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x + C.$$

Dividing both sides by 2, we obtain

$$\int e^x \cos x dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + \frac{1}{2}C.$$

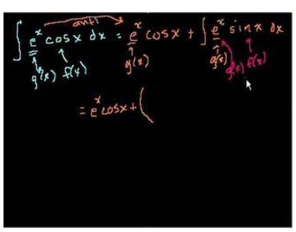
Since the constant of integration is just a "dummy" constant, let $\frac{C}{2} \to C$.

Finally, our solution is

$$\int e^x \cos x dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.$$

Multimedia Links

To see this same "classic" example worked out with narration **17.0**, see Khan Academy Indefinite Integration Series Part 7 (9:38).



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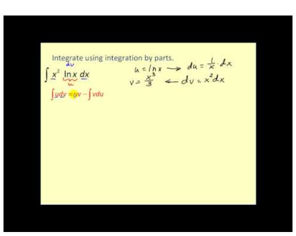
For additional video presentations on integration by parts **17.0**, see Math Video Tutorials by James Sousa, Integration by Parts, Basic (7:08)



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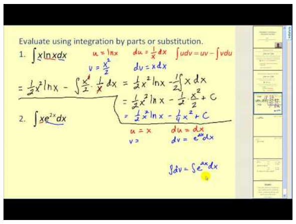
; Math Video Tutorials by James Sousa, Integration by Parts (10:03)



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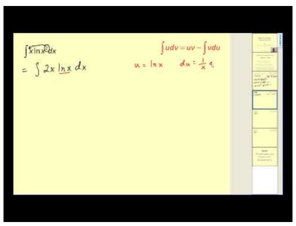
; Math Video Tutorials by James Sousa, Integration by Parts, Additional Examples (7:48).



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Review Questions

Evaluate the following integrals. (*Remark:* Integration by parts is not necessarily a requirement to solve the integrals. In some, you may need to use $u-$substitution along with integration by parts.)

1. $\int 3xe^x dx$
2. $\int x^2 e^{-x} dx$
3. $\int \ln(3x+2) dx$
4. $\int \sin^{-1} x dx$
5. $\int \sec^3 x dx$
6. $\int 2x \ln(3x) dx$
7. $\int \frac{(lnx)^2}{x} dx$
8. Use both the method of $u-$substitution and the method of integration by parts to integrate the integral below. Both methods will produce equivalent answers.

$$\int x\sqrt{5x-2}dx$$

9. Use the method of tabular integration by parts to solve $\int x^2 e^{5x} dx$.
10. Evaluate the definite integral $\int_0^1 x^2 e^x dx$.
11. Evaluate the definite integral $\int_1^3 \ln(x+1) dx$.

7.3 Integration by Partial Fractions

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Integration by Partial Fractions.
- Combine the technique of partial fractions with $u-$substitution to solve various integrals.

This is the third technique that we will study. This technique involves decomposing a rational function into a sum of two or more simple rational functions. For example, the rational function

$$\frac{x+4}{x^2+x-2}$$

can be decomposed into

$$\frac{x+4}{x^2+x-2} = \frac{2}{x+2} + \frac{3}{x-1}.$$

The two partial sums on the right are called *partial factions.* Suppose that we wish to integrate the rational function above. By decomposing it into two partial fractions, the integral becomes manageable:

$$\begin{aligned}\int \frac{x+4}{x^2+x-2}dx &= \int \left(\frac{2}{x+2} + \frac{3}{x-1}\right)\\ &= 2\int \frac{1}{x+1}dx + 3\int \frac{1}{x-1}dx\\ &= 2\ln|x+1| + 3\ln|x-1| + C.\end{aligned}$$

To use this method, we must be able to factor the denominator of the original function and then decompose the rational function into two or more partial fractions. The examples below illustrate the method.

Example 1:

Find the partial fraction decomposition of

$$\frac{2x-19}{x^2+x-6}.$$

Solution:

We begin by factoring the denominator as $x^2+x-6 = (x+3)(x-2)$. Then write the partial fraction decomposition as

$$\frac{2x-19}{x^2+x-6}=\frac{A}{x+3}+\frac{B}{x-2}.$$

Our goal at this point is to find the values of A and B. To solve this equation, multiply both sides of the equation by the factored denominator $(x+3)(x-2)$. This process will produce the *basic equation.*

$$2x-19=A(x-2)+B(x+3).$$

This equation is true for all values of x. The most convenient values are the ones that make a factor equal to zero, namely, $x=2$ and $x=-3$. Substituting $x=2$,

$$\begin{aligned}2(2)-19&=A(2-2)+B(2+3)\\-15&=0+5B\\-3&=B\end{aligned}$$

Similarly, substituting for $x=-3$ into the basic equation we get

$$\begin{aligned}2(-3)-19&=A(-3-2)+B(-3+3)\\-25&=-5A+0\\5&=A\end{aligned}$$

We have solved the basic equation by finding the values of A and B. Therefore, the partial fraction decomposition is

$$\frac{2x-19}{x^2+x-6}=\frac{5}{x+3}-\frac{3}{x-2}.$$

General Description of the Method

To be able to write a rational function $f(x)/g(x)$ as a sum of partial fractions, we must apply two conditions:

- The degree of $f(x)$ must be less than the degree of $g(x)$. If so, the rational function is called *proper*. If it is not, divide $f(x)$ by $g(x)$ (use long division) and work with the remainder term.
- The factors of $g(x)$ are known. If not, you need to find a way to find them. The guide below shows how you can write $f(x)/g(x)$ as a sum of partial fractions if the factors of $g(x)$ are known.

A Guide to Finding Partial Fractions Decomposition of a Rational Function

1. To find the partial fraction decomposition of a proper rational function, $f(x)/g(x)$, factor the denominator $g(x)$ and write an equation that has the form

$$\frac{f(x)}{g(x)}=(\text{sum of partial fractions.})$$

2. For each distinct factor $ax+b$, the right side must include a term of the form

$$\frac{A}{ax+b}.$$

3. For each repeated factor $(ax+b)^n$, the right side must include n terms of the form

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}.$$

Example 2:

Use the method of partial fractions to evaluate $\int \frac{x+1}{(x+2)^2}dx$.

Solution:

According to the guide above (item #3), we must assign the sum of $n=2$ partial sums:

$$\frac{x+1}{(x+2)^2}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}.$$

Multiply both sides by $(x+2)^2$:

$$x+1=A(x+2)+B$$
$$x+1=Ax+(2A+B).$$

Equating the coefficients of like terms from both sides,

$$1=A$$
$$1=2A+B.$$

Thus

$$A=1.$$
$$B=-1.$$

Therefore the partial fraction decomposition is

$$\frac{x+1}{(x+2)^2}=\frac{1}{x+2}-\frac{1}{(x+2)^2}.$$

The integral will become

$$\begin{aligned}\int \frac{x+1}{(x+2)^2}dx &= \int \left(\frac{1}{x+2} - \frac{1}{(x+2)^2}\right) \\ &= \int \frac{1}{x+2}dx - \int \frac{1}{(x+2)^2}dx \\ &= \ln\left|x+2\right| + \frac{1}{x+2} + C,\end{aligned}$$

where we have used $u-$substitution for the second integral.

Example 3:

Evaluate $\int \frac{3x^2+3x+1}{x^3+2x^2+x}dx$.

Solution:

We begin by factoring the denominator as $x(x+1)^2$. Then the partial fraction decomposition is

$$\frac{3x^2+3x+1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying each side of the equation by $x(x+1)^2$ we get the basic equation

$$3x^2+3x+1 = A(x+1)^2 + Bx(x+1) + Cx.$$

This equation is true for all values of x. The most convenient values are the ones that make a factor equal to zero, namely, $x = -1$ and $x = 0$.

Substituting $x = -1$,

$$\begin{aligned}3(-1)^2+3(-1)+1 &= A(-1+1)^2 + B(-1)(-1+1) + C(-1) \\ 1 &= 0+0-C \\ -1 &= C.\end{aligned}$$

Substituting $x = 0$,

$$\begin{aligned}3(0)^2+3(0)+1 &= A(0+1)^2 + B(0)(0+1) + C(0) \\ 1 &= A+0+0 \\ 1 &= A.\end{aligned}$$

To find B we can simply substitute any value of x along with the values of A and C obtained.

Choose $x = 1$:

$$\begin{aligned}3(1)^2+3(1)+1 &= A(1+1)^2 + B(1)(1+1) + C(1) \\ 7 &= 4+2B-1 \\ 2 &= B.\end{aligned}$$

Now we have solved for A, B, and C. We use the partial fraction decomposition to integrate.

$$\begin{aligned}\int \frac{3x^2+3x+1}{x^3+2x^2+x}dx. &= \int \left(\frac{1}{x}+\frac{2}{x+1}-\frac{1}{(x+1)^2}\right)dx \\ &= \ln|x|+2\ln|x+1|+\frac{1}{x+1}+C.\end{aligned}$$

Example 4:

This problem is an example of an improper rational function. Evaluate the definite integral

$$\int_1^2 \frac{x^3-4x^2-3x+3}{x^2-3x}dx.$$

Solution:

This rational function is improper because its numerator has a degree that is higher than its denominator. The first step is to divide the denominator into the numerator by long division and obtain

$$\frac{x^3-4x^2-3x+3}{x^2-3x} = (x-1)+\frac{-6x+3}{x^2-3x}.$$

Now apply partial function decomposition only on the remainder,

$$\frac{-6x+3}{x^2-3x} = \frac{-6x+3}{x(x-3)} = \frac{A}{x}+\frac{B}{x-3}.$$

As we did in the previous examples, multiply both sides by $x(x-3)$ and then set $x=0$ and $x=3$ to obtain the basic equation

$$-6x+3 = A(x-3)+Bx$$

For $x=0$,

$$\begin{aligned}3 &= -3A+0 \\ -1 &= A.\end{aligned}$$

For $x=3$,

$$\begin{aligned}-18+3 &= 0+3B \\ -15 &= 3B \\ -5 &= B.\end{aligned}$$

Thus our integral becomes

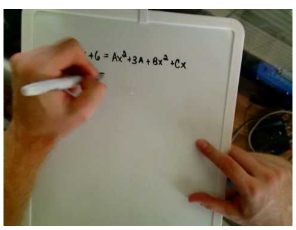
$$\int_1^2 \frac{x^3-4x^2-3x+3}{x^2-3x}dx = \int_1^2 \left[(x-1)+\frac{-6x+3}{x^2-3x}\right]dx = \int_1^2 \left[x-1-\frac{1}{x}-\frac{5}{x-3}\right]dx.$$

Integrating and substituting the limits,

$$\begin{aligned}
&= \left[\frac{x^2}{2}-x-\ln|x|-5\ln|x-3|\right]_1^2 \\
&= \left(\frac{4}{2}-2-\ln 2-5\ln 1\right)-\left(\frac{1}{2}-1-\ln 1-5\ln 2\right) \\
&= 4\ln 2+\frac{1}{2}.
\end{aligned}$$

Multimedia Links

For a complete partial fractions problem **(19.0)**, see Integration by Partial Fractions, Just Math Tutoring (6:02)



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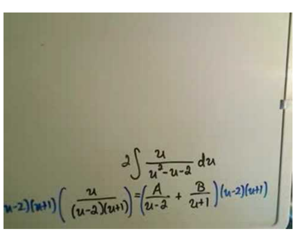
and for integration using partial fractions and a rationalizing substitution **(19.0)**, see Integration using Partial Fractions and a rationalizing substitution, Just Math Tutoring (6:06).



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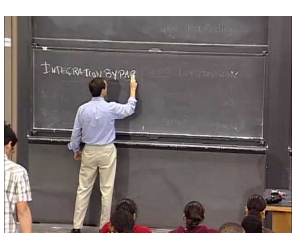
For an extensive presentation on integrating with partial fractions including the completing the square technique **(19.0)** see Integration with partial fractions using various techniques, MIT Courseware (51:24).



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Review Questions

Evaluate the following integrals.

1. $\int \frac{1}{x^2-1} dx$
2. $\int \frac{x}{x^2-2x-3} dx$
3. $\int \frac{1}{x^3+x^2-2x} dx$
4. $\int \frac{x^3}{x^2+4} dx$
5. $\int_0^1 \frac{\phi}{1+\phi} d\phi$
6. $\int_1^5 \frac{x-1}{x^2(x+1)} dx$
7. Evaluate the integral by making the proper $u-$substitution to convert to a rational function: $\int \frac{cos\theta}{\sin^2\theta+4\sin\theta-5} d\theta$.
8. Evaluate the integral by making the proper $u-$substitution to convert to a rational function: $\int \frac{3e^\theta}{e^{2\theta}-1} d\theta$.
9. Find the area under the curve $y = 1/(2+e^x)$, over the interval $[-\ln 3, \ln 4]$. (*Hint*: make a $u-$substitution to convert the integrand into a rational function.)
10. Show that $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$.

7.4 Trigonometric Integrals

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the Trigonometric Integrals.
- Combine this technique with $u-$substitution.

Integrating Powers of Sines and Cosines

In this section we will study methods of integrating functions of the form

$$\int \sin^m x \cos^n x dx,$$

where m and n are nonnegative integers. The method that we will describe uses the famous trigonometric identities
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

Example 1:

Evaluate $\int \sin^2 x dx$ and $\int \cos^2 x dx$.

Solution:

Using the identities above, the first integral can be written as

$$\begin{aligned}\int \sin^2 x dx &= \int \frac{1}{2}(1-\cos 2x)dx \\ &= \frac{1}{2}\int (1-\cos 2x)dx \\ &= \frac{1}{2}(x-\frac{1}{2}\sin 2x)+C \\ &= \frac{x}{2}-\frac{1}{4}\sin 2x+C.\end{aligned}$$

Similarly, the second integral can be written as

$$\begin{aligned}\int \cos^2 x dx &= \int \frac{1}{2}(1+\cos 2x)dx \\ &= \frac{1}{2}\int (1+\cos 2x)dx \\ &= \frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C \\ &= \frac{x}{2}+\frac{1}{4}\sin 2x+C.\end{aligned}$$

Example 2:

Evaluate $\int \cos^4 x dx$.

Solution:

$$\begin{aligned}\int \cos^4 x dx = \int (\cos^2 x)^2 dx &= \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx \\ &= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)dx \\ &= \frac{1}{4}\int \left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right) dx \\ &= \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right) dx.\end{aligned}$$

Integrating term by term,

$$\begin{aligned}&= \frac{1}{4}\left[\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right]+C \\ &= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C.\end{aligned}$$

Example 3:

Evaluate $\int \sin^3 x dx$.

Solution:

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

Recall that $\sin^2 x+\cos^2 x = 1$, so by substitution,

$$\begin{aligned}&= \int (1-\cos^2 x)\sin x dx \\ &= \int \sin x dx - \int \cos^2 x \sin x dx.\end{aligned}$$

The first integral should be straightforward. The second can be done by the method of $u-$substitution by letting $u = \cos x$, so $du = -\sin x dx$. The integral becomes

$$\begin{aligned}
&= -\cos x - \int \left[-u^2 \sin x \frac{du}{\sin x} \right] \\
&= -\cos x + \int u^2 du \\
&= -\cos x + \frac{u^3}{3} + C \\
&= -\cos x + \frac{1}{3}\cos^3 x + C.
\end{aligned}$$

If m and n are both positive integers, then an integral of the form

$$\int \sin^m x \cos^n x dx$$

can be evaluated by one of the procedures shown in the table below, depending on whether m and n are odd or even.

TABLE 7.2:

$\int \sin^m x \cos^n x dx$	Procedure	Identities
n odd	Let $u = \sin x$	$\cos^2 x = 1 - \sin^2 x$
m odd	Let $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
n and m even	Use identities to reduce powers	$\sin^2 x = (1/2)(1 - \cos 2x)$
		$\cos^2 x = (1/2)(1 + \cos 2x)$

Example 4:

Evaluate $\int \sin^3 x \cos^4 x dx$.

Solution:

Here, m is odd. So according to the second procedure in the table above, let $u = \cos x$, so $du = -\sin x$. Substituting,

$$\begin{aligned}
\int \sin^3 x \cos^4 x dx &= \int u^4 \sin^3 x \frac{-1}{\sin x} du \\
&= -\int u^4 \sin^2 x du.
\end{aligned}$$

Referring to the table again, we can now substitute $\sin^2 x = 1 - \cos^2 x$ in the integral:

$$\begin{aligned}
&= -\int u^4 (1 - \cos^2 x) du \\
&= -\int u^4 (1 - u^2) du \\
&= \int (-u^4 + u^6) du \\
&= \frac{-1}{5} u^5 + \frac{1}{7} u^7 + C \\
&= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.
\end{aligned}$$

Example 5:

Evaluate $\int \sin^4 x \cos^4 x dx$.

Solution:

Here, $m = n = 4$. We follow the third procedure in the table above:

$$\begin{aligned}\int \sin^4 x \cos^4 x dx &= \int (\sin^2 x)^2 (\cos^2 x)^2 dx \\ &= \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \left[\frac{1}{2}(1 + \cos 2x)\right]^2 dx \\ &= \frac{1}{16} \int (1 - \cos^2 2x)^2 dx \\ &= \frac{1}{16} \int \sin^4 2x dx.\end{aligned}$$

At this stage, it is best to use $u-$substitution to integrate. Let $u = 2x$, so $du = 2dx$.

$$\begin{aligned}\int \sin^4 x \cos^4 x dx &= \frac{1}{32} \int \sin^4 u du \\ &= \frac{1}{32} \int (\sin^2 u)^2 du = \frac{1}{32} \int \left[\frac{1}{2}(1 - \cos 2u)\right]^2 du \\ &= \frac{1}{32} \left(\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right) + C \\ &= \frac{3}{256}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C.\end{aligned}$$

Integrating Powers of Secants and Tangents

In this section we will study methods of integrating functions of the form

$$\int \tan^m x \sec^n x dx,$$

where m and n are nonnegative integers. However, we will begin with the integrals

$$\int \tan x dx$$

and

$$\int \sec x dx.$$

The first integral can be evaluated by writing

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Using $u-$substitution, let $u = \cos x$, so $du = -\sin x dx$. The integral becomes

$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{u} \frac{-1}{\sin x}\\ du &= -\int \frac{1}{u} du = -\ln|u| + C\\ &= -\ln|\cos x| + C\\ &= \ln(1/|\cos x|) + C\\ &= \ln|\sec x| + C.\end{aligned}$$

The second integral $\int \sec x dx$, however, is not straightforward—it requires a trick. Let

$$\begin{aligned}\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx\\ &= \int \frac{sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.\end{aligned}$$

Use $u-$substitution. Let $u = \sec x + \tan x$, then $du = (\sec^2 x + \sec x \tan x)dx$, the integral becomes,

$$\begin{aligned}\int \sec x dx &= \int \frac{du}{u}\\ &= \ln|u| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

There are two reduction formulas that help evaluate higher powers of tangent and secant:

$$\begin{aligned}\int \sec^n x dx &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx,\\ \int \tan^m x dx &= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x dx.\end{aligned}$$

Example 6:

Evaluate $\int \sec^3 x dx$.

Solution:

We use the formula above by substituting for $n = 3$.

$$\begin{aligned}\int \sec^3 x dx &= \frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1} \int \sec x dx\\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x dx\\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.\end{aligned}$$

Example 7:

Evaluate $\int \tan^5 x dx$.

Solution:

We use the formula above by substituting for $m = 5$.

$$\int \tan^5 x dx = \frac{\tan^4 x}{4} - \int \tan^3 x dx.$$

We need to use the formula again to solve the integral $\int \tan^3 x dx$:

$$\begin{aligned} \int \tan^5 x dx &= \frac{\tan^4 x}{4} - \int \tan^3 x dx \\ &= \frac{\tan^4 x}{4} - \left[\frac{\tan^2 x}{2} - \int \tan x dx \right] \\ &= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C. \end{aligned}$$

If m and n are both positive integers, then an integral of the form

$$\int \tan^m x \sec^n x dx$$

can be evaluated by one of the procedures shown in the table below, depending on whether m and n are odd or even.

TABLE 7.3:

$\int \tan^m x \sec^n x dx$	Procedure	Identities
n even	Let $u = \tan x$	$\sec^2 x = \tan^2 x + 1$
m odd	Let $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
m even	Reduce powers of $\sec x$	$\tan^2 x = \sec^2 x - 1$
n odd		

Example 8:

Evaluate $\int \tan^2 x \sec^4 x dx$.

Solution:

Here $n = 4$ is even, and so we will follow the first procedure in the table above. Let $u = \tan x$, so $du = \sec^2 x dx$. Before we substitute, split off a factor of $\sec^2 x$.

$$\int \tan^2 x \sec^4 x dx = \int \tan^2 x \sec^2 x \sec^2 x dx.$$

Since $\sec^2 x = \tan^2 x + 1$,

$$= \int \tan^2 x (\tan^2 x + 1) \sec^2 x dx.$$

Now we make the $u-$substitution:

$$\begin{aligned}
&= \int u^2(u^2+1)du \\
&= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C.
\end{aligned}$$

Example 9:

Evaluate $\int \tan^3 x \sec^3 x dx$.

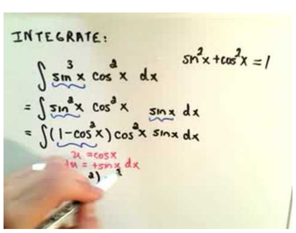
Solution:

Here $m = 3$ is odd. We follow the second procedure in the table. Make the substitution, $u = \sec x$ and $du = \sec x \tan x dx$. Our integral becomes

$$\begin{aligned}
\int \tan^3 x \sec^3 x dx &= \int \tan^2 x \sec^2 x(\sec x \tan x) dx \\
&= \int (\sec^2 x - 1)\sec^2 x(\sec x \tan x)dx \\
&= \int (u^2-1)u^2 du \\
&= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\
&= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.
\end{aligned}$$

Multimedia Links

For video presentations on computing the integrals of trigonometric functions **(20.0)**, see Trigonometric Integrals, Part 1 (5:57)



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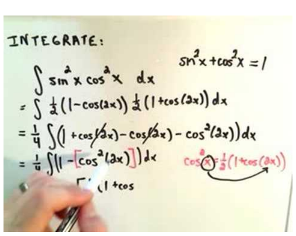
; see Trigonometric Integrals, Part 2 (6:01)



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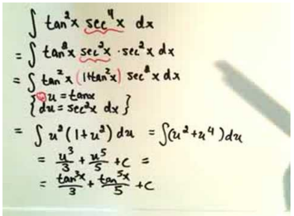
; see Trigonometric Integrals, Part 3 (5:54)



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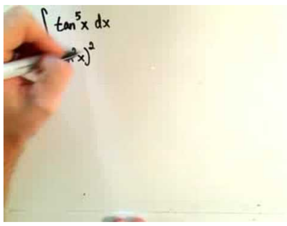
; see Trigonometric Integrals, Part 4 (8:57)



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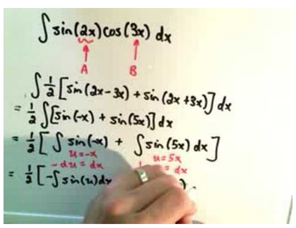
; see Trigonometric Integrals, Part 5 (5:58)



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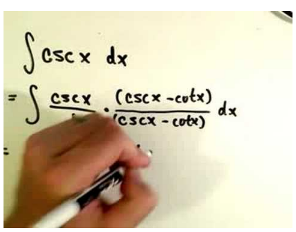
; see Trigonometric Integrals, Part 6 (8:42).



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Review Questions

Evaluate the integrals.

1. $\int \cos^4 x \sin x dx$
2. $\int \sin^2 5\phi d\phi$
3. $\int \sin^2 2z \cos^3 2z dz$
4. $\int \sin x \cos(x/2) dx$
5. $\int \sec^4 x \tan^3 x dx$

6. $\int \tan^4 x \sec x dx$
7. $\int \sqrt{\tan x} \sec^4 x dx$
8. $\int_0^{\pi/2} \tan^5 \frac{x}{2} dx$
9. Graph and then find the volume of the solid that results when the region enclosed by $y = \sin x, y = \cos x, x = 0$, and $x = \pi/4$ is revolved around the $x-$axis.
10.
    a. Prove that $\int \csc x dx = -\ln|\csc x + \cot x| + C$
    b. Show that it can also be written in the following two forms: $\int \csc x dx = \ln|\tan \frac{1}{2} x| + C = \ln|\csc x - \cot x| + C$.

7.5 Trigonometric Substitutions

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Trigonometric Substitution.
- Combine this technique with other integration techniques to integrate.

When we are faced with integrals that involve radicals of the forms $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, and $\sqrt{x^2+a^2}$, we may make substitutions that involve trigonometric functions to eliminate the radical. For example, to eliminate the radical in the expression

$$\sqrt{a^2-x^2}$$

we can make the substitution

$$x = a\sin\theta,$$

$$-\pi/2 \le \theta \le \pi/2,$$

(Note: θ must be limited to the range of the inverse sine function.)

which yields,

$$\begin{aligned}\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} &= \sqrt{a^2(1-\sin^2\theta)}\\ = a\sqrt{\cos^2\theta} = a\cos\theta.\end{aligned}$$

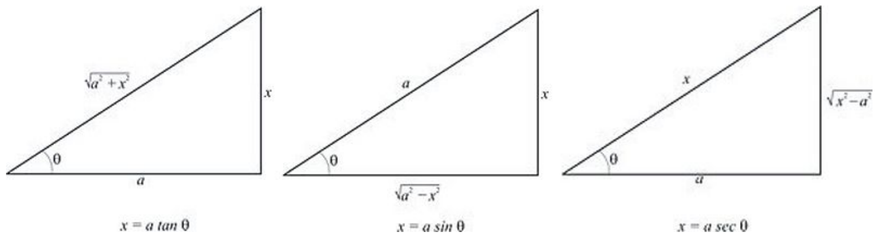
The reason for the restriction $-\pi/2 \le \theta \le \pi/2$ is to guarantee that $\sin\theta$ is a one-to-one function on this interval and thus has an inverse.

The table below lists the proper trigonometric substitutions that will enable us to integrate functions with radical expressions in the forms above.

TABLE 7.4:

Expression in Integrand	Substitution	Identity Needed
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$sec^2\theta - 1 = \tan^2\theta$

In the second column are listed the most common substitutions. They come from the reference right triangles, as shown in the figure below. We want any of the substitutions we use in the integration to be reversible so we can change back to the original variable afterward. The right triangles in the figure below will help us reverse our substitutions.



Description: 3 triangles.

Example 1:

Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution:

Our goal first is to eliminate the radical. To do so, look up the table above and make the substitution

$$x = 2\sin\theta, -\pi/2 \le \theta \le \pi/2,$$

so that

$$\frac{dx}{d\theta} = 2\cos\theta$$

Our integral becomes

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos\theta d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}} \\ &= \int \frac{2\cos\theta d\theta}{(2\sin\theta)^2(2\cos\theta)} \\ &= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} \\ &= \frac{1}{4}\int \csc^2\theta d\theta \\ &= -\frac{1}{4}\cot\theta + C.\end{aligned}$$

Up to this stage, we are done integrating. To complete the solution however, we need to express $\cot\theta$ in terms of x. Looking at the figure of triangles above, we can see that the second triangle represents our case, with $a = 2$. So $x = 2\sin\theta$ and $2\cos\theta = \sqrt{4-x^2}$, thus

$$\cot\theta = \frac{\sqrt{4-x^2}}{x},$$

since

$$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

so that

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4-x^2}} &= -\frac{1}{4}\cot\theta + C \\ &= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.\end{aligned}$$

Example 2:

Evaluate $\int \frac{\sqrt{x^2-3}}{x}dx$.

Solution:

Again, we want to first to eliminate the radical. Consult the table above and substitute $x = \sqrt{3}\sec\theta$. Then $dx = \sqrt{3}\sec\theta\tan\theta d\theta$. Substituting back into the integral,

$$\begin{aligned}\int \frac{\sqrt{x^2-3}}{x}dx &= \int \frac{\sqrt{3\sec^2\theta - 3}}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta d\theta \\ &= \sqrt{3}\int \tan^2\theta d\theta.\end{aligned}$$

Using the integral identity from the section on Trigonometric Integrals,

$$\int \tan^m x dx = \frac{\tan^{m-1}x}{m-1} - \int \tan^{m-2}x dx.$$

and letting $m = 2$ we obtain

$$\int \frac{\sqrt{x^2-3}}{x}dx = \sqrt{3}((\tan\theta) - \theta) + C.$$

Looking at the triangles above, the third triangle represents our case, with $a = \sqrt{3}$. So $x = \sqrt{3}\sec\theta$ and thus $\cos x = \sqrt{3}/x$, which gives $\tan\theta = \sqrt{x^2-3}/\sqrt{3}$. Substituting,

$$\begin{aligned}\int \frac{\sqrt{x^2-3}}{x}dx &= \sqrt{3}((\tan\theta) - \theta) + C \\ &= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \tan^{-1}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}}\right)\right) + C \\ &= \sqrt{x^2-3} - \sqrt{3}\tan^{-1}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}}\right) + C.\end{aligned}$$

Example 3:

Evaluate $\int \frac{dx}{x^2\sqrt{x^2+1}}$.

Solution:

From the table above, let $x = \tan\theta$ then $dx = \sec^2\theta d\theta$. Substituting into the integral,

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}.$$

But since $\tan^2\theta + 1 = \sec^2\theta$,

$$\begin{aligned}
&= \int \frac{\sec^2\theta d\theta}{\tan^2\theta \sec\theta} \\
&= \int \frac{\sec\theta}{\tan^2\theta} d\theta \\
&= \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta} d\theta \\
&= \int \cot\theta \csc\theta d\theta.
\end{aligned}$$

Since $\frac{d}{d\theta}(\csc\theta) = -\cot\theta\csc\theta$,

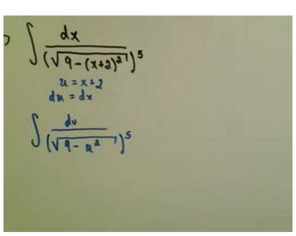
$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+1}} &= \int \cot\theta\csc\theta d\theta \\
&= -\csc\theta + C.
\end{aligned}$$

Looking at the triangles above, the first triangle represents our case, with $a = 1$. So $x = \tan\theta$ and thus $\sin x = \frac{x}{\sqrt{1+x^2}}$, which gives $\csc\theta = \frac{\sqrt{1+x^2}}{x}$. Substituting,

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2+1}} &= -\csc\theta + C \\
&= -\frac{\sqrt{1+x^2}}{x} + C.
\end{aligned}$$

Multimedia Links

For video presentations on Trigonometric Substitutions **(17.0)**, see Trigonometric Substitutions, Just Math Tutoring (9:30)



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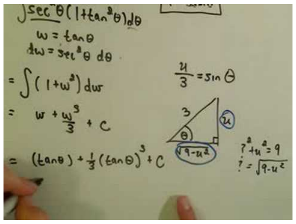
and Trigonometric Substitutions, Part 2, Just Math Tutoring (4:20).



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Review Questions

Evaluate the integrals.

1. $\int \sqrt{4-x^2}dx$
2. $\int \frac{1}{\sqrt{9+x^2}}dx$
3. $\int \frac{x^3}{\sqrt{1-x^2}}dx$
4. $\int \frac{1}{\sqrt{1-9x^2}}dx$
5. $\int x^3 \sqrt{4-x^2}dx$
6. $\int \frac{1}{x^2 \sqrt{x^2-36}}dx$
7. $\int \frac{1}{(x^2+25)^2}dx$
8. $\int_0^4 x^3 \sqrt{16-x^2}dx$
9. $\int_{-\pi}^0 e^x \sqrt{1-e^{2x}}dx$ (*Hint:* First use $u-$substitution, letting $u = e^x$)
10. Graph and then find the area of the surface generated by the curve $y = x^2$ from $x = 1$ to $x = 0$ and revolved about the $x-$axis.

7.6 Improper Integrals

Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using the technique of Improper Integration.
- Combine this technique with other integration techniques to integrate.
- Distinguish between proper and improper integrals.

The concept of ***improper integrals*** is an extension to the concept of definite integrals. The reason for the term *improper* is because those integrals either

- include integration over infinite limits or
- the integrand may become infinite within the limits of integration.

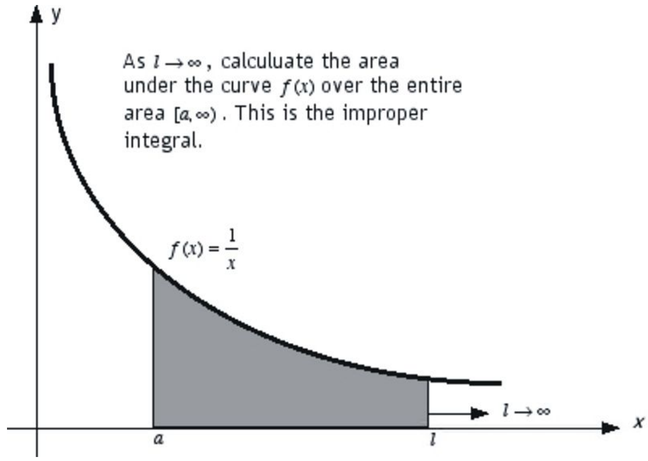
We will take each case separately. Recall that in the definition of definite integral $\int_a^b f(x)dx$ we assume that the interval of integration $[a,b]$ is finite and the function f is continuous on this interval.

Integration Over Infinite Limits

If the integrand f is continuous over the interval $[a,\infty)$, then the improper integral in this case is defined as

$$\int_a^\infty f(x)dx = \lim_{l\to\infty}\int_a^l f(x)dx.$$

If the integration of the improper integral exists, then we say that it ***converges***. But if the limit of integration fails to exist, then the improper integral is said to ***diverge.*** The integral above has an important geometric interpretation that you need to keep in mind. Recall that, geometrically, the definite integral $\int_a^b f(x)dx$ represents the area under the curve. Similarly, the integral $\int_a^l f(x)dx$ is a definite integral that represents the area under the curve $f(x)$ over the interval $[a,l]$, as the figure below shows. However, as l approaches ∞, this area will expand to the area under the curve of $f(x)$ and over the entire interval $[a,\infty)$. Therefore, the improper integral $\int_a^\infty f(x)dx$ can be thought of as the area under the function $f(x)$ over the interval $[a,\infty)$.



Example 1:

Evaluate $\int_1^\infty \frac{dx}{x}$.

Solution:

We notice immediately that the integral is an improper integral because the upper limit of integration approaches infinity. First, replace the infinite upper limit by the finite limit l and take the limit of l to approach infinity:

$$\begin{aligned}\int_1^\infty \frac{dx}{x} &= \lim_{l\to\infty}\int_1^l \frac{dx}{x}\\ &= \lim_{l\to\infty}[\ln x]_1^l\\ &= \lim_{l\to\infty}(\ln l - \ln 1)\\ &= \lim_{l\to\infty}\ln l\\ &= \infty.\end{aligned}$$

Thus the integral diverges.

Example 2:

Evaluate $\int_2^\infty \frac{dx}{x^2}$.

Solution:

$$\begin{aligned}\int_2^\infty \frac{dx}{x^2} &= \lim_{l\to\infty}\int_2^l \frac{dx}{x^2}\\ &= \lim_{l\to\infty}\left[\frac{-1}{x}\right]_2^l\\ &= \lim_{l\to\infty}\left(\frac{-1}{l}+\frac{1}{2}\right)\\ &= \frac{1}{2}.\end{aligned}$$

Thus the integration converges to $\frac{1}{2}$.

Example 3:

Evaluate $\int_{+\infty}^{-\infty} \frac{dx}{1+x^2}$.

Solution:

What we need to do first is to split the integral into two intervals $(-\infty, 0]$ and $[0, +\infty)$. So the integral becomes

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}.$$

Next, evaluate each improper integral separately. Evaluating the first integral on the right,

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{l \to -\infty} \int_{l}^{0} \frac{dx}{1+x^2} \\
&= \lim_{l \to -\infty} \left[\tan^{-1} x\right]_l^0 \\
&= \lim_{l \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} l\right] \\
&= \lim_{l \to -\infty} \left[0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{2}.
\end{aligned}$$

Evaluating the second integral on the right,

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{l \to \infty} \int_{0}^{l} \frac{dx}{1+x^2} \\
&= \lim_{l \to \infty} \left[\tan^{-1} x\right]_0^l \\
&= \frac{\pi}{2} - 0 = \frac{\pi}{2}.
\end{aligned}$$

Adding the two results,

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Remark: In the previous example, we split the integral at $x = 0$. However, we could have split the integral at any value of $x = c$ without affecting the convergence or divergence of the integral. The choice is completely arbitrary. This is a famous thoerem that we will not prove here. That is,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx.$$

Integrands with Infinite Discontinuities

This is another type of integral that arises when the integrand has a vertical asymptote (an infinite discontinuity) at the limit of integration or at some point in the interval of integration. Recall from Chapter 5 in the Lesson on Definite Integrals that in order for the function f to be integrable, it must be bounded on the interval $[a, b]$. Otherwise, the function is not integrable and thus does not exist. For example, the integral

$$\int_0^4 \frac{dx}{x-1}$$

develops an infinite discontinuity at $x = 1$ because the integrand approaches infinity at this point. However, it is continuous on the two intervals $[0,1)$ and $(1,4]$. Looking at the integral more carefully, we may split the interval $[0,4] \to [0,1) \cup (1,4]$ and integrate between those two intervals to see if the integral converges.

$$\int_0^4 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^4 \frac{dx}{x-1}.$$

We next evaluate each improper integral. Integrating the first integral on the right hand side,

$$\begin{aligned} \int_0^1 \frac{dx}{x-1} &= \lim_{l \to 1^-} \int_0^l \frac{dx}{x-1} \\ &= \lim_{l \to 1^-} [\ln|x-1|]_0^l \\ &= \lim_{l \to 1^-} [\ln|l-1| - \ln|-1|] \\ &= -\infty. \end{aligned}$$

The integral diverges because $\ln(0)$ is undefined, and thus there is no reason to evaluate the second integral. We conclude that the original integral diverges and has no finite value.

Example 4:

Evaluate $\int_1^3 \frac{dx}{\sqrt{x-1}}$.

Solution:

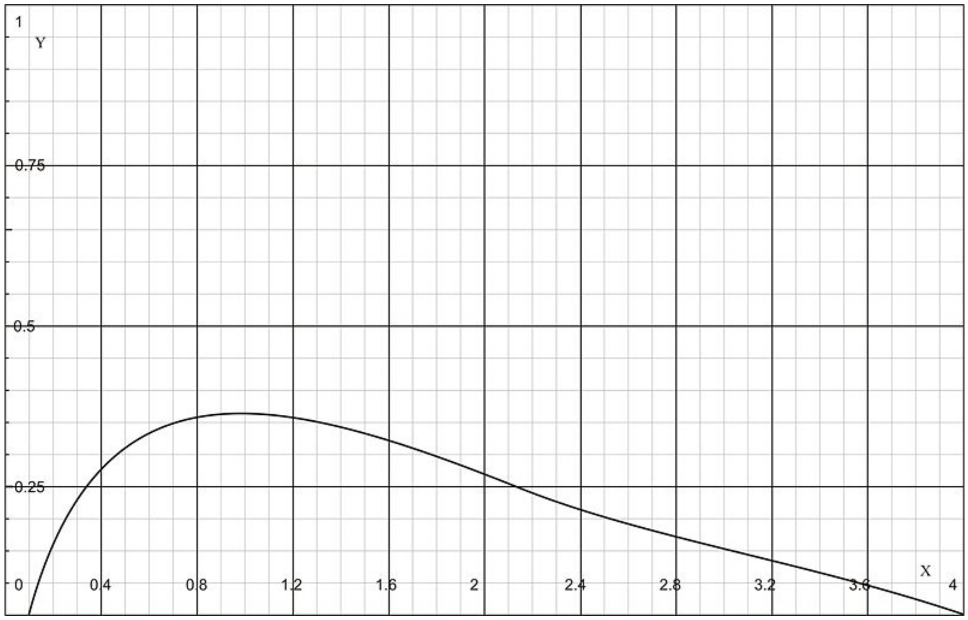
$$\begin{aligned} \int_1^3 \frac{dx}{\sqrt{x-1}} &= \lim_{l \to 1^+} \int_l^3 \frac{dx}{\sqrt{x-1}} \\ &= \lim_{l \to 1^+} \left[2\sqrt{x-1}\right]_l^3 \\ &= \lim_{l \to 1^+} \left[2\sqrt{2} - 2\sqrt{l-1}\right] \\ &= 2\sqrt{2}. \end{aligned}$$

So the integral converges to $2\sqrt{2}$.

Example 5:

In Chapter 5 you learned to find the volume of a solid by revolving a curve. Let the curve be $y = xe^{-x}, 0 \leq x \leq \infty$ and revolving about the $x-$axis. What is the volume of revolution?

Solution:



From the figure above, the area of the region to be revolved is given by $A = \pi y^2 = \pi x^2 e^{-2x}$. Thus the volume of the solid is

$$V = \pi \int_0^\infty x^2 e^{-2x} dx = \pi \lim_{l \to \infty} \int_0^l x^2 e^{-2x} dx.$$

As you can see, we need to integrate by parts twice:

$$\begin{aligned}\int x^2 e^{-2x} dx &= -\frac{x^2}{2} e^{-2x} + \int x e^{-2x} dx \\ &= -\frac{x^2}{2} e^{-2x} - \frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C.\end{aligned}$$

Thus

$$\begin{aligned}V &= \pi \lim_{l \to \infty} \left[-\frac{x^2}{2} e^{-2x} - \frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right]_0^l \\ &= \pi \lim_{l \to \infty} \left[\frac{2x^2 + 2x + 1}{-4e^{2x}} \right]_0^l \\ &= \pi \lim_{l \to \infty} \left[\frac{2l^2 + 2l + 1}{-4e^{2l}} - \frac{1}{-4e^0} \right] \\ &= \pi \lim_{l \to \infty} \left[\frac{2l^2 + 2l + 1}{4e^{2l}} + \frac{1}{4} \right].\end{aligned}$$

At this stage, we take the limit as l approaches infinity. Notice that the when you substitute infinity into the function, the denominator of the expression $\frac{2l^2+2l+1}{-4e^{2l}}$, being an exponential function, will approach infinity at a much faster rate than will the numerator. Thus this expression will approach zero at infinity. Hence

$$V = \pi \left[0 + \frac{1}{4} \right] = \frac{\pi}{4},$$

So the volume of the solid is $\pi/4$.

Example 6:

Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{e^x+e^{-x}}$.

Solution:

This can be a tough integral! To simplify, rewrite the integrand as

$$\frac{1}{e^x+e^{-x}} = \frac{1}{e^{-x}(e^{2x}+1)} = \frac{e^x}{e^{2x}+1} = \frac{e^x}{1+(e^x)^2}.$$

Substitute into the integral:

$$\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x}{1+(e^x)^2} dx.$$

Using $u-$substitution, let $u = e^x, du = e^x dx$.

$$\begin{aligned}\int \frac{dx}{e^x+e^{-x}} &= \int \frac{du}{1+u^2} \\ &= \tan^{-1} u + C \\ &= \tan^{-1} e^x + C.\end{aligned}$$

Returning to our integral with infinite limits, we split it into two regions. Choose as the split point the convenient $x = 0$.

$$\int_{-\infty}^{+\infty} \frac{dx}{e^x+e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^x+e^{-x}} + \int_{0}^{+\infty} \frac{dx}{e^x+e^{-x}}.$$

Taking each integral separately,

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{e^x+e^{-x}} &= \lim_{l\to-\infty} \int_{l}^{0} \frac{dx}{e^x+e^{-x}} \\ &= \lim_{l\to-\infty} \left[\tan^{-1} e^x\right]_l^0 \\ &= \lim_{l\to-\infty} \left[\tan^{-1} e^0 - \tan^{-1} e^l\right] \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4}.\end{aligned}$$

Similarly,

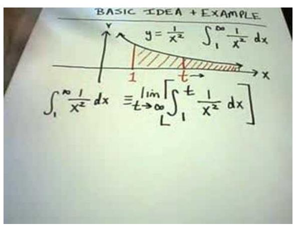
$$\begin{aligned}\int_0^{+\infty} \frac{dx}{e^x+e^{-x}} &= \lim_{l\to\infty}\int_0^{1} \frac{dx}{e^x+e^{-x}}\\ &= \lim_{l\to\infty}\left[\tan^{-1}e^x\right]_0^l\\ &= \lim_{l\to\infty}\left[\tan^{-1}e^l-\tan^{-1}1\right]\\ &= \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.\end{aligned}$$

Thus the integral converges to

$$\int_{-\infty}^{+\infty} \frac{dx}{e^x+e^{-x}} = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}.$$

Multimedia Links

For a video presentation of Improper Integrals **(22.0)**, see Improper Integrals, www.justmathtutoring.com (6:23).



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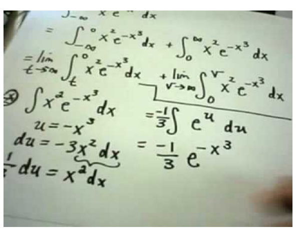
For a video presentation of Improper Integrals with Infinity in the Upper and Lower Limits **(22.0)**, see Improper Integrals, www.justmathtutoring.com (7:55).



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Review Questions

1. Determine whether the following integrals are improper. If so, explain why.
   a. $\int_1^7 \frac{x+2}{x-3}dx$
   b. $\int_1^7 \frac{x+2}{x+3}dx$
   c. $\int_0^1 \ln x dx$
   d. $\int_0^\infty \frac{1}{\sqrt{x-2}}dx$

e. $\int_0^{\pi/4} \tan x dx$

Evaluate the integral or state that it diverges.

2. $\int_1^{\infty} \frac{1}{x^{2.001}} dx$
3. $\int_{-\infty}^{-2} \left[\frac{1}{x-1} - \frac{1}{x+1}\right] dx$
4. $\int_{-\infty}^{0} e^{5x} dx$
5. $\int_3^5 \frac{1}{(x-3)^4} dx$
6. $\int_{-\pi/2}^{\pi/2} \tan x dx$
7. $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$
8. The region between the $x-$axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the $x-$axis.

    a. Find the volume of revolution, V.
    b. Find the surface area of the volume generated, S.

7.7 Ordinary Differential Equations

General and Particular Solutions

Differential equations appear in almost every area of daily life including science, business, and many others. We will only consider *ordinary differential equations* (ODE). An ODE is a relation on a function y of one independent variable x and the derivatives of y with respect to x, i.e. $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$. For example, $y'' + (y')^2 + y = x$.

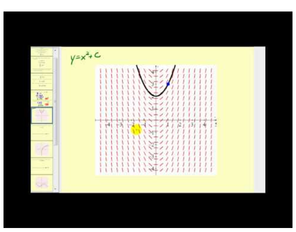
An ODE is *linear* if F can be written as a linear combination of the derivatives of y, i.e. $y^{(n)} = \sum a_i(x)y(i) + r(x)$. A linear ODE is *homogeneous* if $r(x) = 0$.

A *general solution* to a linear ODE is a solution containing a number (the order of the ODE) of arbitrary variables corresponding to the constants of integration. A *particular solution* is derived from the general solution by setting the constants to particular values. For example, for linear ODE of second degree $y'' + y = 0$, a general solution has the form $y_g = A\cos x + B\sin x$ where A, B are real numbers. By setting $A = 1$ and $B = 0, y_p = \cos x$

It is generally hard to find the solution of differential equations. Graphically and numerical methods are often used. In some cases, analytical method works, and in the best case, y has an explicit formula in x.

Multimedia Links

For a video introduction to differential equations **(27.0)**, see Math Video Tutorials by James Sousa, Introduction to Differential Equations (8:12).



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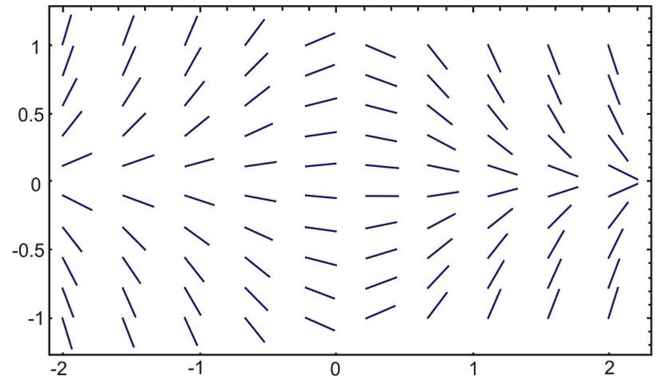
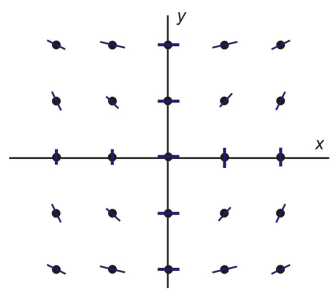
Slope Fields and Isoclines

We now only consider linear ODE of the first degree, i.e. $\frac{dy}{dx} = F(x, y)$. In general, the solutions of a differential equation could be visualized before trying an analytic method. A *solution curve* is the curve that represents a solution (in the $xy-$ plane).

The *slope field* of the differential equation is the set of all short line segments through each point (x, y) and with slope $F(x, y)$.

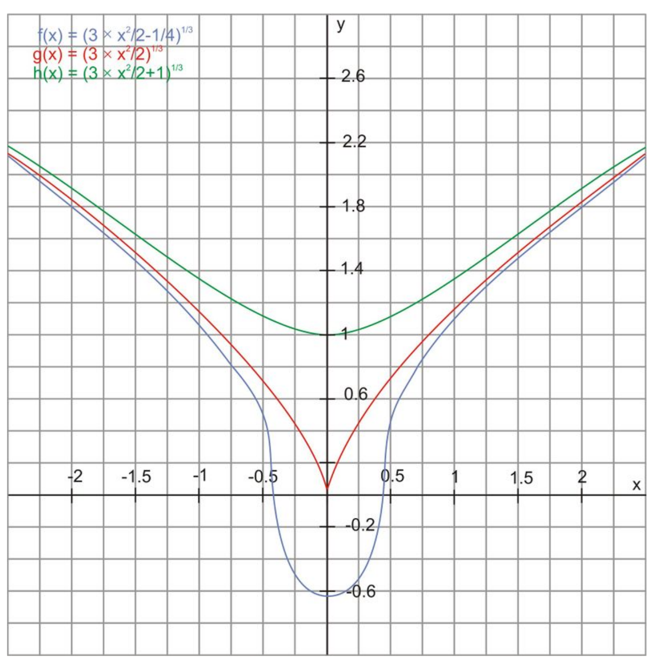


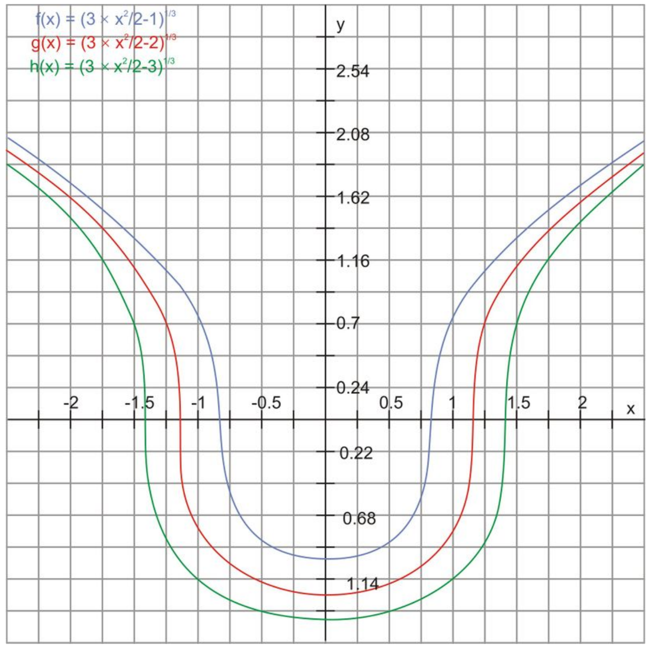


An *isocline* (for constant k) is the line along which the solution curves have the same gradient (k). By calculating this gradient for each isocline, the slope field can be visualized; making it relatively easy to sketch approxi- mate solution curves. For example,$\frac{dy}{dx} = \frac{x}{y}$. The isoclines are $y = \frac{x}{k}$.

Example 1 Consider $\frac{dy}{dx} = \frac{x}{y^2}$. We briefly sketch the slope field as above.

The solutions are $y^3 = \frac{3}{2}\, x^2 + C$.





Exercise

1. Sketch the slope field of the differential equation $\frac{dy}{dx} = 1 - y$. Sketch the solution curves based on it.
2. Sketch the slope field of the differential equation $\frac{dy}{dx} = y - x$. Find the isoclines and sketch a solution curve that passes through $(1,0)$.

Differential Equations and Integration

We begin the analytic solutions of differential equations with a simple type where $F(x,y)$ is a function of x only. $\frac{dy}{dx} = f(x)$ is a function of x. Then any antiderivative of f is a solution by the Fundamental Theorem of Calculus: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Example 1 Solve the differential equation $\frac{dy}{dx} = x$ with $y(0) = 1$.

Solution. $y = \int x\,dx = \frac{x^2}{2} + C$. Then $y(0) = 1$ gives $1 = 0 + C$, i.e. $C = 1$ Therefore $y = \frac{x^2}{2} + 1$.

Exercise

1. Solve the differential equation $\frac{dy}{dx} = \sqrt{9 - x^2}$ with $y(0) = 3$.

Solving Separable First-Order Differential Equations

The next type of differential equation where analytic solution are relatively easy is when the dependence of $F(x,y)$ on x and y are separable: $\frac{dy}{dx} = F(x,y)$ where $F(x,y) = f(x)g(y)$ is the product of a functions of x and y respectively. The solution is in the form $P(x) = Q(y)$. Here $g(y)$ is never 0 or the values of y in the solutions will be restricted by where $g(y) = 0$.

Example 1 Solve the differential equation $y' = xy$ with the initial condition $y(0) = 1$.

Solution. Separating x and y turns the equation in differential form $\frac{dy}{y} = xdx$. Integrating both sides, we have $\ln|y| = \frac{1}{2}x^2 + C$.

Then $y(0) = 1$ gives $\ln|1| = \frac{1}{2}(0)^2 + C$, i.e. $C = 0$ and $\ln|y| = \frac{1}{2}x^2$.

So $|y| = e^{\frac{1}{2}x^2}$.

Therefore, the solutions are $y = \pm e^{\frac{1}{2}x^2}$.

Here $Q(y) = y$ is 0 when $y = 0$ and the values of y in the solutions satisfy $y > 0$ or $y < 0$.

Example 2. Solve the differential equation $2xy' = 1 - y^2$.

Solution. Separating x and y turns the equation in differential form $\frac{2}{1-y^2}dy = \frac{dx}{x}$.

Resolving the partial fraction $\frac{2}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$ gives linear equations $A + B = 2$ and $A - B = 0$.

So $\left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy = \frac{dx}{x}$. Integrating both sides, we have $-\ln|1-y| + \ln|1+y| = \ln|x| + C$ or $\ln\left|\frac{1+y}{1-y}\right| = \ln(e^C|x|) = \ln D|x|$ with $D = e^C > 0$. Then $\left|\frac{1+y}{1-y}\right| = D|x|$, i.e. $\frac{1+y}{1-y} = \pm Dx$ where $D > 0$.

Therefore, the solution has form $y = \pm\frac{Dx-1}{Dx+1}$ where $D > 0$.

Exercise

1. Solve the differential equation $\frac{dy}{dx} = \frac{1}{e^y}$ which satisfies the condition $y(e) = 0$.
2. Solve the differential equation $\frac{dy}{dx} = x(y^2 + 1)$.
3. Solve the differential equation $\frac{dy}{dx} = \frac{x}{\sqrt{1-y^2}}$.

Exponential and Logistic Growth

In some models, the population grows at a rate proportional to the current population without restrictions. The population is given by the differential equation $\frac{dP}{dt} = kP$, where $k > 0$ is the growth rate. In a refined model, the rate of growth is adjusted by another factor $\left(1 - \frac{P}{K}\right)$ where K is the *carrier capacity.* This is close to 1 when P is small compared with K but close to 0 when P is close to K.

Both differential equations are separable and could be solved as in last section. $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$. The solutions are respectively:

$P(t) = P(0)e^{kt}$ and $P(t) = \frac{P_0}{1+Ae^{kt}}$ with $A = \frac{K-P_0}{P_0}$.

Example 1 (Exponential Growth) The population of a group of immigrants increased from 10000 to 20000 from the end of the first year to the end of second year they came to an island. Assuming an exponential growth model on the population, estimate the size of the group of initial immigrants.

Solution. The population of the group is given by $P = P_0e^{kt}$ where the initial population and relative growth rate are to be determined.

At $t = 1$ (year), $P = 10000$, so $10000 = P_0e^{k\cdot 1} = P_0^{e^k}$.

At $t = 2$ (year), $P = 20000$, so $20000 = P_0e^{k\cdot 2} = P_0e^{2k}$.

Dividing both sides of the second equation by the first, we have $2 = e^k$.

Then back in the first equation, $10000 = P_0(2)$. So $P_0 = 5000$. There are 5000 initial immigrants.

Example 2 (Logistic Growth) The population on an island is given by the equation $\frac{dP}{dt} = 0.05P\left(1 - \frac{P_0}{5000}\right), P_0 = 5000$. Find the population sizes $P(20), P(30)$. At what time will the population first exceed 4000?

Solution. The solution is given by $P = \frac{P_0}{1-Ae^{0.05t}}$ where $A = \frac{5000-1000}{1000} = 4$.

$$P(20) = \frac{5000}{1+4e^{-0.05(20)}} = \frac{5000}{1+4e^{-1}} = 2023$$
$$P(30) = \frac{5000}{1+4e^{-0.05(30)}} = \frac{5000}{1+4e^{-1.5}} = 3785.$$

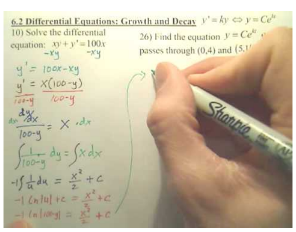
Solve for time, $4000 = \frac{5000}{1+4e^{-0.05(t)}}$ gives $e^{(-0.05t)} = \frac{\frac{5000}{4000}-1}{4} = 0.0625$. So $t = 56$. The population first exceed 4000 in the 56^{th} year.

Exercise

1. (Exponential Growth) The population of a suburban city increased from 10000 in 2005 to 30000 in 2007. Assuming an exponential growth model on the population, by which year will the population first exceed 100000?
2. (Logistic Growth) The population of a city is given by the equation $\frac{dP}{dt} = 0.06P\left(1-\frac{P_0}{100000}\right), P_0 = 25000$. Find the population sizes $P(10), P(25)$. At what time will the population first exceed 90000?

Multimedia Links

For a video presentation of Differential Equations including growth and decay **(27.0)**, see Differential Equations, Growth and Decay (7:23).



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Numerical Methods (Euler's, Improved Euler, Runge-Kutta)

The Euler's method is a numerical approximation to a solution curve starting from the point (a,b) through the algorithm:

$y_{n+1} = y_n + hF(x_n, y_n)$ where $x_0 = a, y_0 = b$ and h is the step size.

The shorter step size, the better is the approximation to the solution curve.

Improved Euler (Heun) method adapts on Euler's method by using both end point values: $y_{n+1} = y_n + \frac{h}{2}[F(x_n, y_n) + F(x_{n+1}, y_{n+1})]$.

Since y_{n+1} also appears on the right side, we replace it by Euler's formula,

$$y_{n+1} = y_n + \frac{h}{2}[F(x_n, y_n) + F(x_{n+1}, y_n + hF(x_n, y_n))].$$

The Runge-Kutta methods are an important family of implicit and explicit iterative methods for the approximation of solutions of our ODE. On them, apply Simpson's rule:

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f'(x)dx = \int_{x_n}^{x_n+h} f'(x)dx$$

$$\approx \frac{h}{6}\left\{y'(x_n) + 4y'\left(x_n + \frac{h}{2}\right) + y'(x_n + 1)\right\}.$$

Exercise 1. Apply the Euler's, improved Euler's and the Runge-Kutta methods on the ODE
$\frac{dy}{dx} = y$ to approximate the solution that satisfy $y(0) = 1$ from $x = 0$ to $x = 1$ with $h = 0.2$.
We know the exact solution is $y = e^x$. Compare their relative accuracy against the exact solution.

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9732 .

CHAPTER 8 Infinite Series

Chapter Outline

This chapter introduces the study of sequences and infinite series. In calculus, we are interested in the behavior of sequences and series, including finding whether a sequence approaches a number or whether an infinite series adds up to a number. The tests and properties in this chapter will help you describe the behavior of a sequence or series.

8.1 Sequences

Learning Objectives

- Demonstrate an understanding of sequences and their terms
- Determine if the limit of a sequence exists and, if it exists, find the limit
- Apply rules, theorems, and Picard's method to compute the limits of sequences

Sequences (rules, terms, indices)

The alphabet, the names in a phone book, the numbered instructions of a model airplane kit, and the schedule in the local television guide are examples of sequences people may use. These examples are all sets of ordered items. In mathematics, a sequence is a list of numbers. You can make finite sequences, such as $2,4,6,8$. These sequences end. You can also make infinite sequences, such as $3,5,7,9,\ldots,$ which do not end but continue on as indicated by the three dots. In this chapter the word *sequence* refers to an infinite sequence.

Each term in a sequence is defined by its place of order in the list. Consider the sequence $3,5,7,9,\ldots$ The first term is 3 because it belongs to place 1 of the sequence. The second term is 5 because it belongs to the second place of the sequence. Likewise, The third term is 7 because it is in the third place. Notice that there is a natural relationship between the counting numbers, or the positive integers, and the terms of the sequence. This leads us to the definition of a **sequence.**

Sequence

A **sequence** is a function from the domain of the set of counting numbers, or positive integers, to the range which consists of the members of a sequence.

A sequence can be denoted by $\{a_n\}$ or by $a_1,a_2,a_3,a_4,\ldots,a_n,\ldots$

The numbers $a_1,a_2,a_3,a_4,\ldots,a_n,\ldots$ that belong to a sequence are called **terms** of the sequence. Each subscript of $1,2,3,\ldots$ on the terms $a_1,a_2,a_3,a_4,\ldots$ refers to the place of the terms in the sequence, or the **index.** The subscripts are called the **indices** of the terms. We assume that $n = 1,2,3,\ldots,$ unless otherwise noted.

Instead of listing the elements of a sequence, we can define a sequence by a **rule,** or formula, in terms of the indices.

Example 1

The formula $a_n = \frac{1}{n}$ is a rule for a sequence.

We can generate the terms for this rule as follows:

n	1	2	3	4	$\ldots$
$a_n = \frac{1}{n}$	$\frac{1}{1} = 1$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\ldots$

Example 2

Consider the sequence rule $a_n = \frac{n^2}{n+1}$.

The terms of the sequence are:

n	1	2	3	4	...
$a_n = \frac{n^2}{n+1}$	$\frac{1^2}{1+1} = \frac{1}{2}$	$\frac{2^2}{2+1} = \frac{4}{3}$	$\frac{9}{4}$	$\frac{16}{5}$	...

You can also find the rule for a sequence.

Example 3

Find the rule for the sequence below.

n	1	2	3	4	...
$a_n = ?$	$\frac{1}{2}$	$-\frac{2}{3}$	$\frac{3}{4}$	$-\frac{4}{5}$	...

Look at each term in terms of its index. The numerator of each term matches the index. The denominator is one more than the index. So far, we can write the formula a_n as $\frac{n}{n+1}$. However, we are not done. Notice that each even-indexed term has a negative sign. This means that all of terms of the sequence have a power of -1. The powers of -1 alternate between odd and even. Usually, alternating powers of -1 can be denote by $(-1)^n$ or $(-1)^{n+1}$. Since the terms are negative for even indices, we use $(-1)^{n+1}$. Thus, the rule for the sequence is $a_n = \frac{(-1)^{n+1}n}{n+1}$. You can check the rule by finding the first few terms of the sequence $a_n = \frac{(-1)^{n+1}n}{n+1}$.

Limit of a Sequence

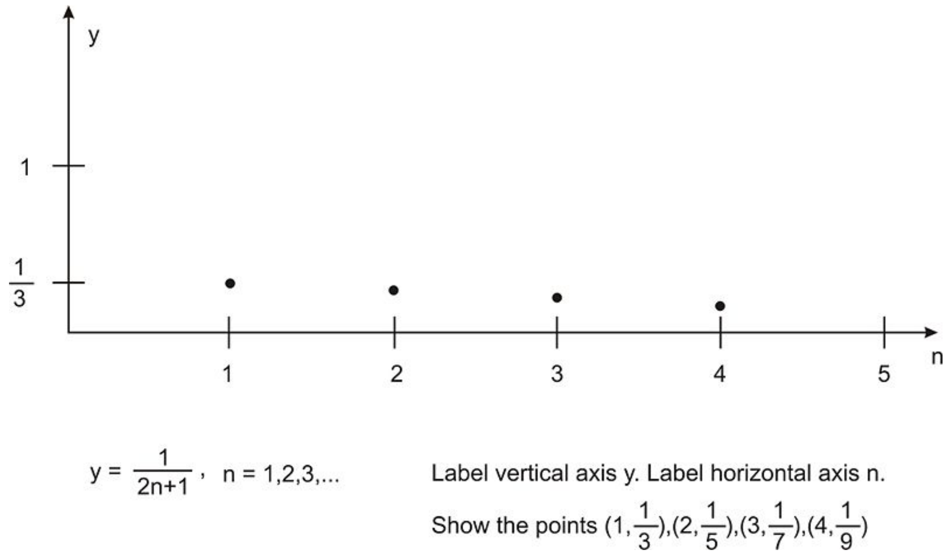
We are interested in the behavior of the sequence as the value of n gets very large. Many times a sequence will get closer to a certain number, or **limit,** as n gets large. Finding the limit of a sequence is very similar to finding the limit of a function. Let's look at some graphs of sequences.

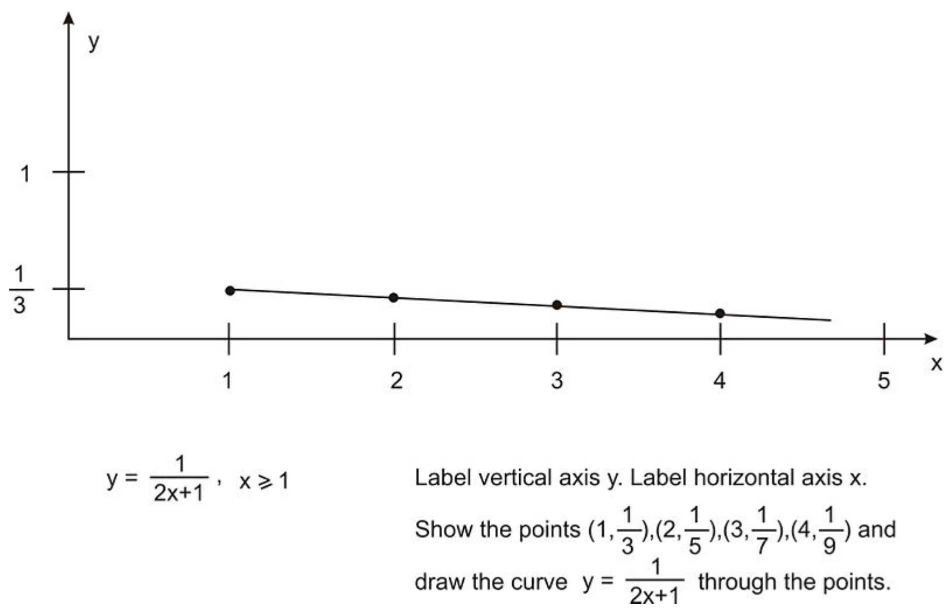
Example 4

Find the limit of the sequence $\left\{\frac{1}{2n+1}\right\}$ as n goes to infinity.

Solution

We can graph the corresponding function $y = \frac{1}{2n+1}$ for $n = 1, 2, 3, \ldots$. The graph of is similar to the continuous function $y = \frac{1}{2x+1}$ for the domain of $x \geq 1$.





To determine the limit, we look at the trend or behavior of the graph of sequence as n gets larger or travels out to positive infinity. This means we look at the points of sequence that correspond to the far right end of the horizontal axis in the figure above. We see that the points of the sequence are getting closer to the horizontal axis, $y = 0$. Thus, the limit of the sequence $\left\{\frac{1}{2n+1}\right\}$ is 0 as n tends to infinity. We write: $\lim_{n\to+\infty} \frac{1}{2n+1} = 0$.

Here is the precise definition of the limit of a sequence.

Limit of a Sequence

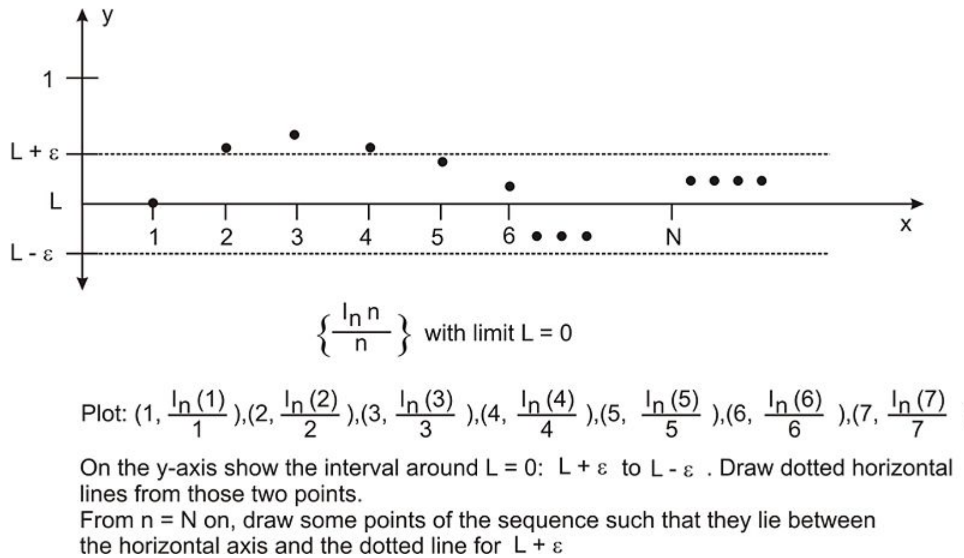
The **limit of a sequence** a_n is the number L if for each $\varepsilon > 0$, there exists an integer N such that $| a_n - L | < \varepsilon$ for all $n > N$.

Recall that $| a_n - L | < \varepsilon$ means the values of a_n such that $L - \varepsilon < a_n < L + \varepsilon$.

What does the definition of the limit of a sequence mean? Here is another example.

Example 5

Look at the figure below.

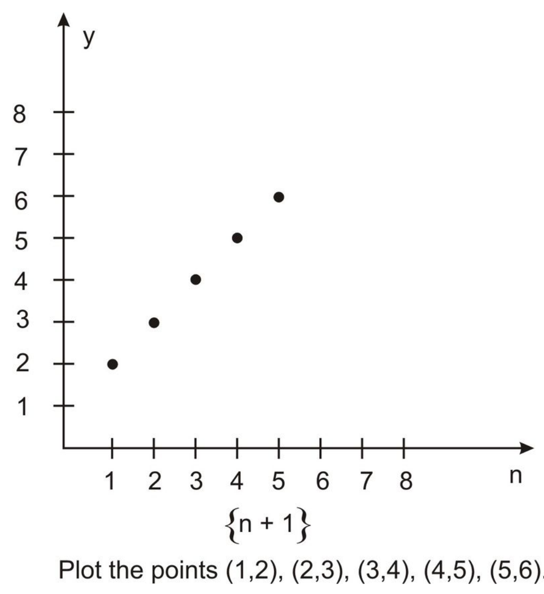


The figure above shows the graph of the sequence $\left\{\frac{l_n\ (n)}{n}\right\}$. Notice that from N on, the terms of $\frac{l_n\ n}{n}$ are between $L-\varepsilon$ and $L+\varepsilon$. In other words, for this value of ε, there is a value N such that all terms of a_n are in the interval from $L-\varepsilon$ and $L+\varepsilon$. Thus, $\lim_{n\to+\infty}\frac{l_n\ (n)}{n}=0$.

Not every sequence has a limit.

Example 6

Here is a graph of the sequence $\{n+1\}$.



Consider the sequence $\{n+1\}$ in the figure above. As n gets larger and goes to infinity, the terms of $a_n = n+1$ become larger and larger. The sequence $\{n+1\}$ does not have a limit. We write $\lim_{n\to+\infty}(n+1) = +\infty$.

Convergence and Divergence

We say that a sequence $\{a_n\}$ **converges** to a limit L if sequence has a finite limit L. The sequence has **convergence.** We describe the sequence as **convergent.** Likewise, a sequence $\{a_n\}$ diverges to a limit L if sequence does not have a finite limit. The sequence has **divergence** and we describe the sequence as **divergent.**

Example 7

The sequence $\{\ln(n)\}$ grows without bound as n approaches infinity. Note that the related function $y = \ln(x)$ grows without bound. The sequence is divergent because it does not have a finite limit. We write $\lim_{n\to+\infty} \ln(n) = +\infty$.

Example 8

The sequence $\left\{4 - \frac{8}{n}\right\}$ converges to the limit $L = 4$ and hence is convergent. If you graph the function $y = 4 - \frac{8}{n}$ for $n = 1, 2, 3, \ldots$, you will see that the graph approaches 4 as n gets larger. Algebraically, as n goes to infinity, the term $-\frac{8}{n}$ gets smaller and tends to 0 while 4 stays constant. We write $\lim_{n\to+\infty} \left(4 - \frac{8}{n}\right) = 4$.

Example 9

Does the sequence s_n with terms $1, -1, 1, -1, 1, -1, \ldots$. have a limit?

Solution

This sequence oscillates, or goes back and forth, between the values 1 and -1. The sequence does not get closer to 1 or -1 as n gets larger. We say that the sequence does not have a limit, or $\lim_{n\to+\infty} s_n$ does not exist.

Note: Each sequence's limit falls under only one of the four possible cases:

1. A limit exists and the limit is L: $\lim_{n\to+\infty} s_n = L$.
2. There is no limit: $\lim_{n\to+\infty} s_n$ does not exist.
3. The limit grows without bound in the positive direction and is divergent: $\lim_{n\to+\infty} s_n = +\infty$.
4. The limit grows without bound in the negative direction and is divergent: $\lim_{n\to+\infty} s_n = -\infty$.

If a sequence has a finite limit, then it only has one value for that limit.

Theorem

If a sequence is convergent, then its limit is unique.

Keep in mind that being divergent is not the same as not having a limit.

L'Hôpital's Rule

Realistically, we cannot graph every sequence to determine if it has a finite limit and the value of that limit. Nor can we make an algebraic argument for the limit for every possible sequence. Just as there are indeterminate forms when finding limits of functions, there are indeterminate forms of sequences, such as $\frac{0}{0}, \frac{\infty}{\infty}, 0 + \infty$. To find the limit of such sequences, we can apply L'Hôpital's rule.

Example 10

Find $\lim_{n\to+\infty} \frac{\ln(n)}{n}$.

Solution

We solved this limit by using a graph in Example 5. Let's solve this problem using L'Hôpital's rule. The numerator is $\ln(n)$ and the denominator is n. Both functions $y = \ln(n)$ and $y = n$ do not have limits. So, the sequence $\left\{\frac{\ln(n)}{n}\right\}$ is of the indeterminate form $\frac{\infty}{\infty}$. Since the functions $y = \ln(n)$ and $y = n$ are not differentiable, we apply L'Hôpital's rule to the corresponding problem, $\lim_{x\to+\infty} \frac{\ln(x)}{x}$, first. Taking the first derivative of the numerator and denominator of $y = \frac{\ln(x)}{x}$, we find $\lim_{x\to+\infty} \frac{\ln(x)}{x} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$. Thus, $\lim_{n\to+\infty} \frac{\ln(n)}{n} = 0$ because the points of $y = \frac{\ln(n)}{n}$ are a subset of the points of the function $y = \frac{\ln(x)}{x}$ as x approaches infinity. We also confirmed the limit of the sequence with its graph in Example 5.

Rules, Sandwich/Squeeze

Properties of function limits are also used with limits of sequences.

Theorem (Rules)

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to+\infty} a_n = L_1$ and $\lim_{n\to+\infty} b_n = L_2$.

Let c be any constant. Then the following statements are true:

1. $\lim_{n\to+\infty} c = c$

The limit of a constant is the same constant.

2. $\lim_{n\to+\infty} c \times a_n = c \times \lim_{n\to+\infty} a_n = cL_1$

The limit of a constant times a sequence is the same as the constant times the limit of the sequence.

3. $\lim_{n\to+\infty}(a_n + b_n) = \lim_{n\to+\infty} a_n + \lim_{n\to+\infty} = L_1 + L_2$

The limit of a sum of sequences is the same as the sum of the limits of the sequences.

4. $\lim_{n\to+\infty}(a_n \times b_n) = \lim_{n\to+\infty} a_n \times \lim_{n\to+\infty} b_n = L_1 L_2$

The limit of the product of sequences is the same as the product of the limits of the sequences.

5. If $L_2 \neq 0$, then $\lim_{n\to+\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n} = \frac{L_1}{L_2}$.

The limit of the quotient of two sequences is the same as the quotient of the limits of the sequences.

Let's apply these rules to help us find limits.

Example 11

Find $\lim_{n\to+\infty} \frac{7n}{9n+5}$.

Solution

We could use L'Hôpital's rule or we could use some of the rules in the preceding theorem. Let's use the rules in the theorem. Divide both the numerator and denominator by the highest power of n in the expression and using rules from the theorem, we find the limit:

$$\begin{aligned}
\lim_{n\to+\infty} \frac{7n}{9n+5} &= \lim_{n\to+\infty} \frac{\frac{7n}{n}}{\frac{9n}{n}+\frac{5}{n}} \text{ Dividing both numerator and denominator by } n \\
&= \lim_{n\to+\infty} \frac{7}{\left(9+\frac{5}{n}\right)} \text{ Simplifying} \\
&= \frac{\lim_{n\to+\infty} 7}{\lim_{n\to+\infty}\left(9+\frac{5}{n}\right)} \text{ Applying the division rule for limits.} \\
&= \frac{\lim_{n\to+\infty} 7}{\lim_{n\to+\infty} 9 + \lim_{n\to+\infty} \frac{5}{n}} \text{ Applying the rule for the limit of a sum to the denominator} \\
&= \frac{7}{9+0} = \frac{7}{9} \text{ Evaluating the limits}
\end{aligned}$$

Example 12

Find $\lim_{n\to+\infty}\left(\frac{11}{n} - \frac{8}{n^2}\right)$.

Solution

$$\begin{aligned}\lim_{n\to+\infty}\left(\frac{11}{n}-\frac{8}{n^2}\right) &= \lim_{n\to+\infty}\frac{11}{n}-\lim_{n\to+\infty}\frac{8}{n^2} \text{ Applying the rule for the difference of two limits}\\ &= 11\lim_{n\to+\infty}\frac{1}{n}-8\lim_{n\to+\infty}\frac{1}{n^2} \text{ Applying the rule for the limit of c times a limit}\\ &= 11\times 0-8\times 0=0 \text{ Evaluating the limits}\end{aligned}$$

As with limits of functions, there is a Sandwich/Squeeze Theorem for the limits of sequences.

Sandwich/Squeeze Theorem

Let $\{a_n\},\{b_n\}$ and $\{c_n\}$ be sequences. Let N be a positive integer.

Suppose c_n is a sequence such that $a_n \le c_n \le b_n$ for all $n \ge N$. Suppose also that

$\lim a_n = \lim b_n = L$. Then $\lim c_n = L$.

You can see how the name of the theorem makes sense from the statement. After a certain point in the sequences, the terms of a sequence c_n are sandwiched or squeezed between the terms of two convergent sequences with the same limit. Then the limit of the sequence c_n is squeezed to become the same as the limit of the two convergent sequences. Let's look at an example.

Example 13

Prove $\lim_{n\to+\infty}\frac{8^n}{n!}=0$.

Solution

Recall that $n!$ is read as "n factorial" and is written as $n! = n\times(n-1)\times(n-2)\times\ldots\times 1$.

We want to apply the Sandwich theorem by squeezing the sequence $\frac{8^n}{n!}$ between two sequences that converge to the same limit.

First, we know that $0 \le \frac{8^n}{n!}$. Now we want to find a sequence whose terms greater than or equal to the terms of the sequence $\frac{8}{7}$ for some n.

We can write

$$\begin{aligned}\frac{8^n}{n!} &= \frac{8\times 8\times 8\ldots\times 8}{n\times(n-1)\times(n-2)\times\ldots 1}\\ &= \frac{8}{n}\times\frac{8}{n-1}\times\ldots\times\frac{8}{2}\times\frac{8}{1}\\ &= \left(\frac{8}{n}\right)\left(\frac{8}{n-1}\times\ldots\times\frac{8}{9}\times\frac{8}{8}\right)\left(\frac{8}{7}\times\frac{8}{6}\times\frac{8}{5}\times\ldots\times\frac{8}{1}\right)\end{aligned}$$

Since each factor in the product $\frac{8}{n-1}\times\ldots\times\frac{8}{9}\times\frac{8}{8}$ is less than or equal to 1, then the product $\frac{8}{n-1}\times\ldots\times\frac{8}{9}\times\frac{8}{8}\le 1$. Then we make an inequality:

$$\begin{aligned}\left(\frac{8}{n}\right)\left(\frac{8}{n-1}\times\ldots\times\frac{8}{9}\times\frac{8}{8}\right)\left(\frac{8}{7}\times\frac{8}{6}\times\frac{8}{5}\times\ldots\times\frac{8}{1}\right) &\le \left(\frac{8}{7}\right)(1)\left(\frac{8}{7}\times\frac{8}{6}\times\frac{8}{5}\times\ldots\times\frac{8}{1}\right)\\ &= \left(\frac{8}{7}\right)\left(\frac{8}{7}\times\frac{8}{6}\times\frac{8}{5}\times\ldots\times\frac{8}{1}\right)\\ &= \left(\frac{8}{n}\right)\left(\frac{8^7}{7!}\right)\end{aligned}$$

Thus, $\lim_{n\to+\infty} 0 \le \lim_{n\to+\infty} \frac{8^n}{n!} \le \lim_{n\to+\infty} \left(\frac{8}{n}\right)\left(\frac{8^7}{7!}\right)$. By using the Rules Theorem, we have $\lim_{n\to+\infty} 0 = 0$ and $\lim_{n\to+\infty} \left(\frac{8}{n}\right)\left(\frac{8^7}{7!}\right) = \left(\frac{8^7}{7!}\right)\lim_{n\to+\infty} \frac{8}{n} = \left(\frac{8^7}{7!}\right) \times 0 = 0$. Thus, $0 \le \lim_{n\to+\infty} \frac{8^n}{n!} \le 0$. By the Sandwich/Squeeze Theorem, $\lim_{n\to+\infty} \frac{8^n}{n!} = 0$.

Picard's Method

The following method appeared in 1891 by Emile Picard, a famous French mathematician. It is a method for solving initial value problems in differential equations that produces a sequence of functions which converge to the solution. Start with the initial value problem:

$y' = f(x,y)$ with $y(x_0) = y_0$

If $f(x,y)$ and $f_x(x,y)$ are both continuous then a unique solution to the initial value problem exists by Picard's theory. Now if $y(x)$ is the solution to the given problem, then a reformulation of the differential equation is possible:

$$\int_{x_0}^{x} y'(t)dt = \int_{x_0}^{x} f(t,y(t))dt$$

Now the Fundamental Theorem of Calculus is utilized to integrate the left hand side of the problem and upon isolating , the following result is obtained:

$$y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))dt$$

The equation above is the starting point for the Picard iteration because it will be used to build the sequence of functions which will describe the actual solution to the initial value problem. The Picard sequence of functions is calculated as follows:

Step 1 - Define $Y_0(x) = y_0$

Step 2 - Substitute $Y_0(t) = y_0$ for $y(t)$ in $f(t,y(t))$:

$$Y_1(x) = y_0 + \int_{x_0}^{x} f(t,Y_0(t))dt$$
$$Y_1(x) = y_0 + \int_{x_0}^{x} f(t,Y_0)dt$$

Step 3 - Repeat step 2 with $Y_1(t)$ for $y(t)$:

$$Y_2(x) = y_0 + \int_{x_0}^{x} f(t,Y_1(t))dt$$

The substitution process is repeated n times and generates a sequence of functions $\{Y_n(x)\}$ which converges to the initial value problem. To summarize this procedure mathematically,

Picard's Method

Let $\{Y_n(x)\}$ be sequence defined successively by,

$Y_n(x) = y_0 + \int_{x_0}^{x} f(t, Y_{n-1}(t))dt$ for $n \geq 0$

The sequence of approximations converges to the solution $y(x)$, i.e.

$$\lim_{n\to\infty} Y_n(x) = y(x).$$

Now that we have defined Picard's method, let's calculate a sequence of functions for an initial value problem.

Example 1

Find the first four functions $\{Y_n(x)\}_{n=0}^{3}$ defined by Picard's method for the solution to the initial value problem $y'(x) = xy(x)$ with $y(-1) = 1$.

Solution

We want to apply the Fundamental Theorem of Calculus to the differential equations so that it is reformulated for use in the Picard method. Thus,

$$\begin{aligned} \int_{-1}^{x} y'(t)dt &= \int_{-1}^{x} ty(t)dt \\ y(x) - y(-1) &= \int_{-1}^{x} ty(t)dt \\ y(x) &= 1 + \int_{-1}^{x} ty(t)dt \end{aligned}$$

Now that the differential equation has been rewritten for Picard's method, we begin the calculations for the sequence of functions. In all cases the first function $Y_0(x)$ is given by the initial condition:

Step 1 - Define $Y_0(x) = 1$

Step 2 - Substitute $Y_0(x) = 1$ for $y(t)$ in the integrand of $y(x) = 1 + \int_{-1}^{x} ty(t)dt$:

$$\begin{aligned} Y_1(x) &= 1 + \int_{-1}^{x} t\,dt \\ Y_1(x) &= 1 + \left.\frac{t^2}{2}\right|_{-1}^{x} \\ Y_1(x) &= \frac{1}{2} + \frac{x^2}{2} \end{aligned}$$

Step 3 - Substitute $Y_1(x) = \frac{1}{2} + \frac{x^2}{2}$ for $y(t)$ in the integrand as above:

$$\begin{aligned} Y_2(x) &= 1 + \int_{-1}^{x} t\left(\frac{1}{2} + \frac{t^2}{2}\right)dt \\ Y_2(x) &= 1 + \left.\left(\frac{t^2}{4} + \frac{t^4}{8}\right)\right|_{-1}^{x} \\ Y_2(x) &= \frac{5}{8} + \frac{x^2}{4} + \frac{x^4}{8} \end{aligned}$$

Step 4 - Substitute $Y_2(x) = \frac{5}{8} + \frac{x^2}{4} + \frac{x^4}{8}$ for $y(t)$ in the integrand as done previously:

$$Y_3(x) = 1 + \int_{-1}^{x} t\left(\frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8}\right) dt$$
$$Y_3(x) = 1 + \left(\frac{5t^2}{16} + \frac{t^4}{16} + \frac{t^6}{48}\right)\Big|_{-1}^{x}$$
$$Y_3(x) = \frac{29}{48} + \frac{5x^2}{16} + \frac{x^4}{16} + \frac{x^6}{48}$$

Thus, the initial four functions in the sequence defined by Picard's method are:

$$\left\{1, \frac{1}{2} + \frac{x^2}{2}, \quad \frac{5}{8} + \frac{x^2}{4} + \frac{x^4}{8}, \quad \frac{29}{48} + \frac{5x^2}{16} + \frac{x^4}{16} + \frac{x^6}{48}\right\}$$

The method also states that this sequence will converge to the solution $y(x)$ of the initial value problem, i.e.

$$\lim_{n \to +\infty} Y_n(x) = y(x)$$

A pattern of the functions in the sequence $Y_n(x)$ is emerging but it is not an obvious one. We do know $Y_n(x)$ will converge to the solution for this problem by Picard's method. The exact solution for this problem can be calculated and is given by:

$$y(x) = e^{\frac{x^2-1}{2}}$$

Clearly this solution satisfies $y(x) = xy(x)$ and $y(-1) = 1$.

Review Questions

1. Find the rule for the sequence a_n.

n	1	2	3	4	...
$a_n =?$	-2	2	-2	2	...

Tell if each sequence is convergent, is divergent, or has no limit. If the sequence is convergent, find its limit.

2. $\left\{\frac{4}{n} + \frac{3}{n^2}\right\}$
3. $\left\{6 - \frac{7}{\sqrt{n}}\right\}$
4. $-5, 5, -5, 5, -5, 5, \ldots$
5. $\left\{\frac{4n^6-7}{3n}\right\}$
6. $\left\{\frac{(-1)^n}{5n^2}\right\}$
7. $\{(-1)^n n\}$
8. $\left\{(-1)^n \frac{3n^4-2}{2n^4+6n^2-4n}\right\}$

9. $\left\{\frac{6n^2}{e^n}\right\}$
10. Let $\{a_n\}$ be a sequence such that $\lim_{n\to+\infty}|a_n|=0$. Show that $\lim_{n\to+\infty} a_n = 0$. ($|a_n|$ is the absolute value of a_n.)
11. Find the first four functions $\{Y_n(x)\}_{n=0}^{3}$ defined by Picard's method for the solution to the initial value problem $y'(x) = 1 + y$ with $y(0) = 0$.
12. Find the first four functions $\{Y_n(x)\}_{n=0}^{3}$ defined by Picard's method for the solution to the initial value problem $y'(x) = 1 + y^2$ with $y(0) = 0$.
13. Find the first three functions $\{Y_n(x)\}_{n=0}^{2}$ defined by Picard's method for the solution to the initial value problem $y'(x) = y^{1/3}$ with $y(0) = \frac{1}{8}$.

Keywords

1. sequence
2. rules
3. terms
4. index, indices
5. limit
6. convergence
7. divergence
8. L'Hôpital's Rule
9. Sandwich/Squeeze Theorem
10. Picard's Method

8.2 Infinite Series

Learning Objectives

- Demonstrate an understanding of series and the sequence of partial sums
- Recognize geometric series and determine when they converge or diverge
- Compute the sum of a convergent geometric series
- Determine convergence or divergence of series using the nth-Term Test

Infinite Series (series, sequence of partial sums, convergence, divergence)

Series

Another topic that involves an infinite number of terms is the topic of **infinite series**. We can represent certain functions and numbers with an infinite series. For example, any real number that can be written as a non-terminating decimal can be represented as an infinite series.

Example 1

The rational number $\frac{4}{9}$ can be written as $0.44444\ldots$. We can expand the decimal notation as an infinite series:

$$\begin{aligned}\frac{4}{9} &= 0.4+0.04+0.004+0.0004+\ldots\\ &= \frac{4}{10}+\frac{4}{100}+\frac{4}{1000}+\frac{4}{10,000}+\ldots\\ &= \frac{4}{10}+\frac{4}{10^2}+\frac{4}{10^3}+\frac{4}{10^4}+\ldots\end{aligned}$$

On the other hand, the number $\frac{1}{4}$ can be written as 0.25. If we expand the decimal notation, we get a finite series:

$$\begin{aligned}\frac{1}{4} &= 0.2+0.05\\ &= \frac{2}{10}+\frac{5}{100}\\ &= \frac{2}{10}+\frac{5}{10^2}\end{aligned}$$

Do you see the difference between an infinite series and a finite series? Let's define what we mean by an *infinite series*.

Infinite Series

An infinite series is the sum of an infinite number of terms, $u_1, u_2, u_3, u_4, \ldots$, usually written as.

$$u_1 + u_2 + u_3 + u_4 + \ldots.$$

A shorthand notation for an infinite series is to use sigma notation:

$\sum_{k=1}^{\infty} u_k$, which can be read as "the sum of the terms u_k 's for k equal to 1 to infinity."

We can make finite sums from the terms of the infinite series:

$$\begin{aligned}
s_1 &= u_1 \\
s_2 &= u_1 + u_2 \\
s_3 &= u_1 + u_2 + u_3
\end{aligned}$$

The first sum is the first term of the sequence. The second sum is the sum of the first two terms. The third term is the sum of the first three terms. Thus, the nth finite sum, s_n is the sum of the first n terms of the infinite series: $s_n = u_1 + u_2 + u_3 + \ldots + u_n$.

Sequence of Partial Sums

As you can see, the sums $s_n = u_1 + u_2 + u_3 + \ldots + u_n$ form a sequence. The sequence is very important for the study of the related infinite series for it tells a lot about the infinite series.

Partial Sums

For an infinite series $\sum_{k=1}^{\infty} u_k$, the n **th** partial sum, s_n is the sum of the first n terms of the infinite series: $s_n = \sum_{k=1}^{n} u_k$.

The sequence $\{s_n\}$ formed from these sums is called the **sequence of partial sums**.

Example 2

Find the first five partial sums of the infinite series $1 + 0.1 + 0.01 + 0.001 + \ldots$.

Solution

$$\begin{aligned}
s_1 &= u_1 = 1 \\
s_2 &= u_1 + u_2 = 1 + 0.1 = 1.1 \\
s_3 &= 1 + 0.1 + 0.01 = 1.11 \\
s_4 &= 1 + 0.1 + 0.01 + 0.001 + 0.0001 = 1.111 \\
s_5 &= 1 + 0.1 + 0.01 + 0.001 + 0.0001 = 1.1111
\end{aligned}$$

To further explore series, try experimenting with this applet. The applet shows the terms of a series as well as selected partial sums of the series. Series Applet . As you see from this applet, for some series the partial sums appear to approach a fixed number, while for other series the partial sums do not. Exploring this phenomenon is the topic of the next sections.

Convergence and Divergence

Just as with sequences, we can talk about convergence and divergence of infinite series. It turns out that the convergence or divergence of an infinite series depends on the convergence or divergence of the sequence of partial sums.

Convergence/Divergence of Series

Let $\sum_{k=1}^{\infty} u_k$ be an infinite series and let $\{s_n\}$ be the sequence of partial sums for the series. If $\{s_n\}$ has a finite limit l , then the infinite series converges and $\sum_{k=1}^{\infty} u_k = l$.

If $\{s_n\}$ does not have a finite limit, then the infinite series diverges. The infinite series does not have a sum.

Example 3

Does the infinite series $1+0.1+0.01+0.001+\ldots$ converge or diverge?

Solution

To make our work easier, write the infinite series $1+0.1+0.01+0.001+\ldots$ as an infinite series of fractions:

$$1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots$$

To solve for convergence or divergence of the infinite series, write the formula for the nth partial sum $s_n = \sum_{k=1}^{n} u_k$: $s_n = 1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}$. Note that the nth partial sum ends with a power of $n-1$ in the denominator because 1 is the first term of the infinite series.

It is rather difficult to find $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} 1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}$ as it is written. We will "work" the sum into a different form so that we can find the limit of the sequence of partial sums.

First, multiply both sides of the equation $s_n = 1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}$ by $\frac{1}{10}$:

$$\frac{1}{10}s_n = \frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}\right)$$
$$\frac{1}{10}s_n = \frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots+\frac{1}{10^n}$$

Now we have two equations:

$$s_n = 1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{n-1}}$$
$$\frac{1}{10}s_n = \frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots+\frac{1}{10^n}$$

Subtract the bottom equation from the top equation to cancel terms and simplifying:

$$s_n = 1+\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots+\frac{1}{10^{n-1}}$$
$$-\frac{1}{10}s_n = -\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\ldots+\frac{1}{10^n}\right)$$

$$s_n - \frac{1}{10}s_n = 1-\frac{1}{10^n}$$
$$\frac{9}{10}s_n = 1-\frac{1}{10^n}$$

Solve for s_n by multiplying both sides of the last equation by $\frac{10}{9}$:

$$s_n = \frac{10}{9}(1 - \frac{1}{10^n})$$

Now we find the limit of both sides:

$$\lim_{n\to+\infty} s_n = \lim_{n\to\infty} \frac{10}{9}\left(1 - \frac{1}{10^n}\right)$$
$$\lim_{n\to+\infty} \frac{10}{9}\left(1 - \frac{1}{10^n}\right) = \lim_{n\to\infty} \frac{10}{9} - \lim_{n\to+\infty} \frac{10}{9}\left(\frac{1}{10^n}\right)$$
$$= \frac{10}{9} - 0 = \frac{10}{9}$$

The sum of the infinite series is $\frac{10}{9}$ and so the series converges.

Geometric Series

The **geometric series** is a special kind of infinite series whose convergence or divergence is based on a certain number associated with the series.

Geometric Series

A geometric series is an infinite series written as

$$a + ar + ar^2 + ar^3 + \ldots + ar^{\{i-1\}} + \ldots.$$

In sigma notation, a geometric series is written as $\sum_{k=1}^{\infty} ar^{k-1}$.

The number r is the **ratio** of the series.

Example 4

Here are some examples of geometric series.

TABLE 8.1:

Geometric Series	a	r
$1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^{k-1}} + \ldots$	1	$\frac{1}{4}$
$-\frac{5}{6} + \frac{5}{6^2} - \frac{5}{6^3} + \ldots + \left(-\frac{5}{6}\right)\left(-\frac{1}{6}\right) + \ldots$	$-\frac{5}{6}$	$-\frac{1}{6}$
$1 + 3 + 3^2 + 3^3 + \ldots + 3^{k-1} + \ldots$	1	3

The convergence or divergence of a geometric series depends on r.

Theorem

Suppose that the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ has ratio r.

1. The geometric series converges if $|r| < 1$ and the sum of the series is $\frac{a}{1-r}$.
2. The geometric series diverges if $|r| \geq 1$.

Example 5

Determine if the series $7+\frac{7}{8}+\frac{7}{8^2}+\frac{7}{8^3}+\ldots+\frac{7}{8^{i-1}}+\ldots$ converges or diverges. If it converges, find the sum of the series.

Solution

The series is a geometric series that can be written as $\sum_{k=1}^{\infty} 7\left(\frac{1}{8}\right)^{k-1}$. Then $a=7$ and the ratio $r=\frac{1}{8}$. Because $\left|\frac{1}{8}\right|<1$, the series converges. The sum of the series is $\frac{a}{1-r}=\frac{7}{1-\frac{1}{8}}=\frac{7}{\frac{7}{8}}=8$.

Example 6

Determine if the series $\sum_{k=1}^{+\infty} 9^{k-1}$ converges or diverges. If it converges, find the sum of the series.

Solution

The series is a geometric series with $a=1$ and the ratio $r=9$. Because $|9|>1$, the series diverges.

Example 7

Determine if $\frac{3}{4}+\frac{3}{4^2}+\frac{3}{4^3}+\ldots+\frac{3(-1)^k}{4^k}+\ldots$ converges or diverges. If it converges, find the sum of the series.

Solution

If we rewrite the series in terms of powers of k, the series looks like this:

$$\frac{3(-1)^1}{4^1}+\frac{3(-1)^2}{4^2}+\frac{3(-1)^3}{4^3}+\ldots+\frac{3(-1)^k}{4^k}+\ldots=3\left(-\frac{1}{4}\right)^1+3\left(-\frac{1}{4}\right)^2+\ldots+3\left(-\frac{1}{4}\right)^k+\ldots.$$

It looks like a geometric series with $a=3$ and $r=-\frac{1}{4}$.Since $\left|-\frac{1}{4}\right|=\frac{1}{4}<1$, the series converges.

However, if we write the definition of a geometric series for $a=3$ and $r=-\frac{1}{4}$, the series looks like this:

$$\begin{aligned}\sum_{k=1}^{+\infty} 3\left(-\frac{1}{4}^{k-1}\right) &= 3\left(-\frac{1}{4}\right)^0+3\left(-\frac{1}{4}\right)^1+3\left(-\frac{1}{4}\right)^2+\ldots\\ &= 3+3\left(-\frac{1}{4}\right)^1+3\left(-\frac{1}{4}\right)^2+\ldots\end{aligned}$$

The original problem, $\frac{3(-1)^1}{4^1}+\frac{3(-1)^2}{4^2}+\frac{3(-1)^3}{4^3}+\ldots+\frac{3(-1)^k}{4^k}+\ldots$, does not have the leading term of 3. This does not affect the convergence but will affect the sum of the series. We need to subtract 3 from the sum of the series $3+3\left(-\frac{1}{4}\right)^1+3\left(-\frac{1}{4}\right)^2+\ldots$ to get the sum of $\frac{3(-1)^1}{4^1}+\frac{3(-1)^2}{4^2}+\frac{3(-1)^3}{4^3}+\ldots+\frac{3(-1)^k}{4^k}+\ldots$.

The sum of the series is: $\frac{a}{1-r}-3=\frac{3}{1-\left(-\frac{1}{4}\right)}-3=\frac{3}{\frac{5}{4}}-3=\frac{12}{5}-3=\frac{12}{5}-\frac{15}{5}=-\frac{3}{5}$.

Other Convergent Series

There are other infinite series that will converge.

Example 8

Determine if $\sum_{k=1}^{+\infty}\left(\frac{2}{k}-\frac{2}{k+1}\right)$ converges or diverges. If it converges, find the sum.

Solution

The nth partial sum s_n is:

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\left(\frac{2}{k}-\frac{2}{k+1}\right) \\ &= \left(\frac{2}{1}-\frac{2}{1+1}\right)+\left(\frac{2}{2}-\frac{2}{2+1}\right)+\left(\frac{2}{3}-\frac{2}{3+1}\right)+\ldots+\left(\frac{2}{n}-\frac{2}{n+1}\right) \\ &= \left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\ldots+\left(\frac{2}{n}-\frac{2}{n+1}\right) \end{aligned}$$

We can simplify s_n further. Notice that the first parentheses has $-\frac{2}{2}$ while the second parentheses has $\frac{2}{2}$. These will add up to 0 and cancel out. Likewise, the $-\frac{2}{3}$ and $\frac{2}{3}$

will cancel out. Continue in this way to cancel opposite terms. This sum is a **telescoping sum**, which is a sum of terms that cancel each other out so that the sum will fold neatly like a folding telescope. Thus, we can write the partial sum as

$$s_n = \frac{2}{1}-\frac{2}{n+1} = 2-\frac{2}{n+1}.$$

Then $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty}\left(2-\frac{2}{n+1}\right) = 2$ and $\sum_{k=1}^{+\infty}\left(\frac{2}{k}-\frac{2}{k+1}\right) = 2$.

Other Divergent Series (*n*th-Term Test)

Determining convergence by using the limit of the sequence of partial sums is not always feasible or practical. For other series, it is more useful to apply tests to determine if an infinite series converges or diverges. Here are two theorems that help us determine convergence or divergence.

Theorem (The *n*th-Term Test)

If the infinite series $\sum_{k=1}^{\infty} u_k$ converges, then $\lim_{k\to+\infty} u_{k=0}$

Theorem

If $\lim_{k\to+\infty} u_k \neq 0$ or $\lim_{k\to+\infty} u_k$ does not exist, then the infinite series $\sum_{k=1}^{\infty} u_k$ diverges.

The first theorem tells us that if an infinite series converges, then the limit of the sequence of terms is 0. The converse is not true: If the limit of the sequence of terms is 0, then the series converges. So, we cannot use this theorem as a test of convergence.

The second theorem tells us that if limit of the sequence of terms is not zero, then the infinites series diverges. This gives us the first test of divergence: the n **th-Term Test** or **Divergence Test**. Note that if the test is applied and the limit of the sequence of terms is 0, we cannot conclude anything and must use another test.

Example 9

Determine if $\sum_{k=1}^{\infty}\frac{k}{k+5}$ converges or diverges.

Solution

We can use the *n*th-Term Test to determine if the series diverges. Then we do not have to check for convergence.

$$\lim_{k\to+\infty}\frac{k}{k+5} = \lim_{k\to+\infty}\frac{\frac{k}{k}}{\frac{k+5}{k}} = \lim_{k\to+\infty}\frac{1}{1+\frac{5}{k}} = \frac{\lim_{k\to+\infty} 1}{\lim_{k\to+\infty} 1+\frac{5}{k}} = 1$$

Because $\lim_{k\to+\infty}\frac{k}{k+5} \neq 0$, the series $\sum_{k=1}^{\infty}\frac{k}{k+5}$ diverges.

Example 10

Determine if $\sum_{k=1}^{\infty} \frac{8}{k-3}$ converges or diverges.

Solution

Using the nth-Term Test, $\lim_{k\to+\infty} \frac{8}{k-3}$. Since the limit is 0, we cannot make a conclusion about convergence or divergence.

Rules for Convergent Series, Reindexing

Rules

As with sequences, there are rules for convergent infinite series that help make it easier to determine convergence.

Theorem (Rules for Convergent Series)

1. Suppose $\sum_{k=1}^{\infty} u_k$ and $\sum_{n=1}^{\infty} v_k$ are convergent series with $\sum_{k=1}^{\infty} u_k = S_1$ and $\sum_{k=1}^{\infty} v_k = S_2$.

Then $\sum_{k=1}^{\infty}(u_k+v_k)$ and $\sum_{k=1}^{\infty}(u_k-v_k)$ are also convergent where

$\sum_{k=1}^{\infty}(u_k+v_k) = \sum_{k=1}^{\infty}(u_k)+\sum_{k=1}^{\infty}(v_k) = S_1+S_2$ and $\sum_{k=1}^{\infty}(u_k-v_k) = \sum_{k=1}^{\infty}(u_k)-\sum_{k=1}^{\infty}(v_k) = S_1-S_2$

(The sum or difference of convergent series is also convergent.)

2. Let $c \neq 0$ be a constant.

Suppose $\sum_{k=1}^{\infty} u_k$ converges and $\sum_{k=1}^{\infty} u_k = S$ Then $\sum_{k=1}^{\infty} cu_k$ also converges where.

$$\sum_{k=1}^{\infty} cu_i = c\sum_{k=1}^{\infty} u_k = cS$$

If $\sum_{k=1}^{\infty} u_k$ diverges, then $\sum_{k=1}^{\infty} cu_k$ also diverges.

(Multiplying by a nonzero constant does not affect convergence or divergence.)

Example 10

Find the sum of $\sum_{k=1}^{\infty}\left(\frac{2}{3^{k-1}}+\frac{1}{8^{k-1}}\right)$.

Solution

Using the Rules Theorem, $\sum_{k=1}^{\infty}\left(\frac{2}{3^{k-1}}+\frac{1}{8^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{2}{3^{k-1}}+\sum_{k=1}^{\infty}\frac{1}{8^{k-1}}$.

$\sum_{k=1}^{\infty}\frac{2}{3^{k-1}}$ is a convergent geometric series with $a=2$ and $r=\frac{1}{3}$. Its sum is $\frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$.

$\sum_{k=1}^{\infty}\frac{1}{8^{k-1}}$ is a convergent geometric series with $a=2$ and $r=\frac{1}{8}$. Its sum is $\frac{2}{1-\frac{1}{8}} = \frac{1}{\frac{7}{8}} = \frac{8}{7}$.

Then $\sum_{k=1}^{\infty}\left(\frac{2}{3^{k-1}}+\frac{1}{8^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{2}{3^{k-1}}+\sum_{i=1}^{\infty}\frac{1}{8^{k-1}} = 3+\frac{8}{7} = \frac{29}{7}$

Example 11

Find the sum of $\sum_{k=1}^{\infty} 2\left(\frac{5}{6^{k-1}}\right)$.

Solution

By the rules for constant in infinite series, $\sum_{k=1}^{\infty} 2\left(\frac{5}{6^{k-1}}\right) = 2\sum_{k=1}^{\infty}\frac{5}{6^{k-1}}$. The series $\sum_{k=1}^{\infty}\frac{5}{6^{k-1}}$ is a geometric series with $a=5$ and $r=\frac{1}{6}$. Note that, by the Theorem on convergence of geometric series, this series converges to 6, that is $\sum_{k=1}^{\infty}\frac{5}{6^{k-1}} = \frac{5}{1-\frac{1}{6}} = \frac{5}{\frac{5}{6}} = 6$.

Then $\sum_{k=1}^{\infty} 2\left(\frac{5}{6^{k-1}}\right) = 2\times 6 = 12$.

Adding or subtracting a finite number of terms from an infinite series does not affect convergence or divergence.

Theorem

If $\sum_{k=1}^{\infty} u_k$ converges, then $\sum_{k=1}^{\infty} u_k + (u_1 + u_2 + \ldots + u_m)$ is also convergent.

If $\sum_{k=1}^{\infty} u_k$ converges, then $\sum_{k=1}^{\infty} u_k - (u_1 + u_2 + \ldots + u_m)$ is also convergent.

Likewise, if $\sum_{k=1}^{\infty} u_k$ diverges, then $\sum_{k=1}^{\infty} u_k + (u_1 + u_2 + \ldots + u_m)$ and $\sum_{k=1}^{\infty} u_k - (u_1 + u_2 + \ldots + u_m)$ are also divergent.

For a convergent series, adding or removing a finite number of terms will not affect convergence, but it will affect the sum.

Example 12

Find the sum of $\sum_{k=1}^{\infty} \frac{3}{5^{k-1}} - \left(3 + \frac{3}{5}\right)$.

Solution

$\sum_{k=1}^{\infty} \frac{3}{5^{k-1}}$ is a geometric series with $a = 3$ and $r = \frac{1}{5}$. Its sum is $\frac{15}{4}$

Then $\sum_{k=1}^{\infty} \frac{3}{5^{k-1}} - \left(3 + \frac{3}{5}\right) = \frac{15}{4} - \frac{18}{5} = \frac{3}{20}$

Reindexing

Another property of convergent series is that we can **reindex** a series without changing its convergence. This means we can start the indices of the series with another number other than 1. Keep the terms in order though for reindexing.

Example 13

$\sum_{k=1}^{\infty} \frac{4}{3^{k-1}}$is a convergent geometric series. It can be reindexed by changing the starting position of i and the power of i. The new series is still convergent.

$$\sum_{k=1}^{\infty} \frac{4}{3^{k-1}} = \sum_{k=6}^{\infty} \frac{4}{3^{k-6}}$$

You can check that the series on the right is the same series as the one of the left by writing out the first few terms for each series. Notice that the terms are still in order.

Review Questions

1. Express the number $\frac{1}{11}$ as an infinite series.
2. Find s_1, s_2, s_3 and for $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k}$.
3. Determine if the infinite series $3 + \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots$ converges or diverges.
4. What are the values of a and r for the geometric series $3 + 3(2)^1 + 3(2)^2 + 3(2)^3 + \ldots.$?

Determine if each infinite series converges or diverges. If a series converges, find its sum.

5. $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^{k-1}$
6. $\sum_{k=1}^{+\infty} \left(-\frac{2}{3}\right)^{k-1}$
7. $\sum_{k=1}^{\infty} \frac{k^3}{k^3-5}$
8. $\sum_{k=1}^{\infty} \frac{4^{k+2}}{9^{k-1}}$
9. Find the sum of $\sum_{k=2}^{\infty} \left(\left(-\frac{2}{3}\right)^{k-1} + \frac{1}{5^{k-1}}\right)$.

10. Suppose $\sum_{k=1}^{\infty} U_k$ is a convergent series and $\sum_{k=1}^{\infty} V_k$ is a divergent series. Explain why $\sum_{k=1}^{\infty}(U_k+V_k)$ and $\sum_{k=1}^{\infty}(U_k-V_k)$ both diverge.
11. Give an example of a geometric series whose sum is -3.
12. Give an example of a telescoping sum whose sum is 4.

Keywords

1. infinite series
2. sequence of partial sums
3. convergence
4. divergence
5. geometric series
6. ratio of geometric series
7. *n*th-Term Test
8. reindexing

8.3 Series Without Negative Terms

Learning Objectives

- Demonstrate an understanding of nondecreasing sequences
- Recognize harmonic series, geometric series, and $p-$ series and determine convergence or divergence
- Apply the Comparison Test, the Integral Test, and the Limit Comparison Test

Nondecreasing Sequences

In order to extend our study on infinite series, we must first take a look at a special type of sequence.

Nondecreasing Sequence

A **nondecreasing sequence** $\{S_n\}$ is a sequence of terms that do not decrease:

$S_1 \leq S_2 \leq S_3 \leq \ldots \leq S_n \leq \ldots$. Each term is greater than or equal to the previous term.

Example 1 $5, 10, 15, 20, \ldots$ is a nondecreasing sequence. Each term is greater than the previous term: $5 < 10 < 15 < 20 < \ldots$.

$10,000, 1000, 100, \ldots$ is not a nondecreasing sequence. Each term is less than the previous term: $10,000, > 1000, > 100 \ldots$.

$3, 3, 4, 4, 5, 5, \ldots$ is a nondecreasing sequence. Each term is less than or equal to the previous term: $3 \leq 3 \leq 4 \leq 4 \leq 5 \leq 5 \leq \ldots$.

A discussion about sequences would not be complete without talking about limits. It turns out that certain nondecreasing sequences are convergent.

Theorem

Let $\{S_n\}$ be a nondecreasing sequence: $S_1 \leq S_2 \leq S_3 \leq \ldots \leq S_n \leq \ldots$.

$$\lim_{n\to\infty} S_n$$

1. If there is a constant B such that $S_n \leq B$ for all n, then $\lim_{n\to\infty} S_n$ exists and $\lim_{n\to\infty} S_n = L$ where $L \leq B$.
2. If the constant B does not exist, then $\lim_{n\to\infty} S_n = +\infty$.

The theorem says that a bounded, convergent, nondecreasing sequence has a limit that is less than or equal to the bound. If we cannot find a bound, the sequence diverges.

Example 2 Determine if the sequence $\left\{\frac{n}{6n+5}\right\}$ converges or diverges. If it converges, find its limit.

Solution

Write the first few terms: $\frac{1}{11}, \frac{2}{17}, \frac{3}{23}, \frac{4}{29}, \ldots$. The sequence is nondecreasing. To determine convergence, we see if we can find a constant B such that $\frac{n}{6n+5} \leq B$. If we cannot find such a constant, then the sequence diverges.

If two fractions have the same numerator but different denominators, the fraction with the smaller denominator is the larger fraction. Thus, $\frac{n}{6n+5} \leq \frac{n}{6n} = \frac{1}{6}$. Then $\frac{n}{6n+5} \leq \frac{1}{6}$ and, in fact, $\lim_{n\to\infty} \frac{n}{6n+5} = \frac{1}{6}$.

Series Without Negative Terms (harmonic, geometric, p-series)

There are several special kinds of series with nonnegative terms, i.e., terms that are either positive or zero. We will study the convergence of such series by studying their corresponding sequences of partial sums.

Let's start with the **harmonic series:**

$$\sum_{k=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

The sequence of partial sums look like this:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{1}{2} \\
S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}
\end{aligned}$$

$$\vdots$$

In order for the harmonic series to converge, the sequence of partial sums must converge. The sequence of partial sums of the harmonic series is a nondecreasing sequence. By the previous theorem, if we find a bound on the sequence of partial sums, we can show that the sequence of partial sums converges and, consequently, that the harmonic series converges.

It turns out that the sequence of partial sums cannot be made less than a set constant B. We will omit the proof here, but the main idea is to show that the a selected infinite subset of terms of the sequence of partial sums are greater than a sequence that diverges, which implies that the sequence of partial sums diverge. Hence, the harmonic series is not convergent.

We can also work with **geometric series** whose terms are all non-negative.

Example 3 The geometric series $\sum_{k=1}^{\infty} \left(\frac{3}{2}\right)^{k-1}$ has all non-negative terms. The sequence of partial sums looks like this:

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + \frac{3}{2} \\
S_3 &= 1 + \frac{3}{2} + \frac{9}{4} \\
S_4 &= 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8}
\end{aligned}$$

$\vdots$

Intuitively, we can see that there is no bound on the sequence of partial sums and so the series diverges. This is confirmed by the fact that the ratio of the series, $r = \frac{3}{2}$, tells us that the geometric series does not converge.

Another example of an important series is the $p-$ **series**:

$\sum_{k=1}^{\infty} \frac{1}{K^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} +$, where $p > 0$.

The $p-$series may look like a harmonic series, but it will converge for certain values of p.

Theorem

The $p-$series $\sum_{k=1}^{\infty} \frac{1}{K^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

Example 4 Determine if $\sum_{k=1}^{\infty} \frac{1}{\sqrt{K}}$ converges or diverges.

Solution

Rewrite $\frac{1}{\sqrt{K}}$ as $\frac{1}{K^{\frac{1}{2}}}$ to get $\sum_{k=1}^{\infty} \frac{1}{\sqrt{K}} = \sum_{k=1}^{+\infty} \frac{1}{K^{\frac{1}{2}}}$. The value of p is $\frac{1}{2}$. This is less than 1, which tells us that the series diverges.

Comparison Test

Now that we have studied series without negative terms, we can apply convergence tests made for such series. The first test we will consider is the **Comparison Test.** The name of the test tells us that we will compare series to determine convergence or divergence.

Theorem (The Comparison Test)

Let $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ be series without negative terms. Suppose that $u_1 \leq v_1, u_2 \leq v_2, \ldots, u_i \leq v_i, \ldots$.

1. If $\sum_{k=1}^{\infty} v_k$ converges, then $\sum_{k=1}^{\infty} u_k$ converges.
2. If $\sum_{k=1}^{\infty} u_k$ diverges, then $\sum_{k=1}^{\infty} v_k$ diverges.

In order to use this test, we must check that for each index k, every u_k is less than or equal to v_k. This is the comparison part of the test. If the series with the greater terms, $\sum_{k=1}^{\infty} v_k$, converges, than the series with the lesser terms , $\sum_{k=1}^{\infty} u_k$, converges. If the lesser series diverges, then the greater series will diverge. You can only use the test in the orders given for convergence or divergence. You cannot use this test to say, for example, that if the greater series diverges, than the lesser series also diverges.

Example 5 Determine whether $\sum_{k=1}^{\infty} \frac{1}{K^3+3}$ converges or diverges.

Solution

$\sum_{k=1}^{\infty} \frac{1}{K^3+3}$ looks similar to $\sum_{k=1}^{\infty} \frac{1}{K^3}$, so we will try to apply the Comparison Test. Begin by comparing each term. For each k, $\frac{1}{K^3+3}$ is less than or equal to $\frac{1}{K^3}$, so $\sum_{k=1}^{\infty} \frac{1}{K^3+3} \leq \sum_{k=1}^{\infty} \frac{1}{K^3}$. Since $\sum_{k=1}^{\infty} \frac{1}{K^3}$ is a convergent $p-$series, then, by the Comparison Test, $\sum_{k=1}^{\infty} \frac{1}{K^3+3}$ also converges.

Example 6 Determine whether $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}-5}$ converges or diverges.

Solution

The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}-5}$ is similar to $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$. Using the Comparison Test, $\frac{1}{\sqrt[4]{k}-5} \geq \frac{1}{\sqrt[4]{k}}$ for all k. The series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ diverges since it is a $p-$series with $p = \frac{1}{4}$. By the Comparison Test, $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}-5}$ also diverges.

The Integral Test

Another useful test for convergence or divergence of an infinite series without negative terms is the **Integral Test**. It involves taking the integral of the function related to the formula in the series. It makes sense to use this kind of test for certain series because the integral is the limit of a certain series.

Theorem (The Integral Test)

Let $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} f(k)$ be a series without negative terms. If $f(x)$ is a decreasing, continuous, non-negative function for $x \geq 1$, then:

1. $\sum_{k=1}^{\infty} u_k$ converges if and only if $\int_1^{\infty} f(x)$ converges.
2. $\sum_{k=1}^{\infty} u_k$ diverges if and only if $\int_1^{\infty} f(x)$ diverges.

In the statement of the Integral Test, we assumed that u^k is a function f of k. We then changed that function f to be a continuous function of x in order to evaluate the integral of f. If the integral is finite, then the infinite series converges. If the integral is infinite, the infinite series diverges. The convergence or divergence of the infinite series depends on the convergence or divergence of the corresponding integral.

Example 7 Determine if $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^{\frac{3}{2}}}$ converges or diverges.

Solution

We can use the Integral Test to determine convergence. Write the integral form:

$$\int_1^{\infty} \frac{1}{(2x+1)^{\frac{3}{2}}} dx.$$

Next, evaluate the integral.

$$\int_1^{\infty} \frac{1}{(2x+1)^{\frac{3}{2}}} dx = \lim_{t \to \infty} \int_1^{t} (2x+1)^{-\frac{3}{2}} dx$$

Use the following $u-$substitution to evaluate the integral:

$$u = 2x+1$$
$$du = 2dx$$

Then $\frac{1}{2}\int u^{-\frac{3}{2}} du = \frac{1}{2}\frac{u^{-1/2}}{(-1/2)} = -\frac{1}{u^{\frac{1}{2}}} = -\frac{1}{\sqrt{u}}$.

Thus, $\lim_{t\to\infty} \int_1^{t} (2x+1)^{-\frac{3}{2}} dx = \lim_{t\to\infty} -\frac{1}{\sqrt{2x+1}}]_1^t = \lim_{t\to\infty} \left(-\frac{1}{\sqrt{3}} + -\frac{1}{\sqrt{2t+1}}\right) = -\frac{1}{\sqrt{3}}$.

Since the integral is finite, the series $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^{\frac{3}{2}}}$ converges by the Integral Test.

Limit Comparison Test, Simplified Limit Comparison Test

Another test we can use to determine convergence of series without negative terms is the **Limit Comparison Test**. It is easier to use than the Comparison Test.

Theorem (The Limit Comparison Test)

Suppose $\sum_{k=1}^{\infty} u_k$ is a series without negative terms. Then one of the following will hold.

1. If $\sum_{k=1}^{\infty} v_k$ is a convergent series without negative terms and $\lim_{k\to\infty} \frac{v_k}{u_k}$ is finite, then $\sum_{k=1}^{\infty} u_k$ converges.
2. If $\sum_{k=1}^{\infty} w_k$ is a divergent series without negative terms and $\lim_{k\to\infty} \frac{v_k}{w_k}$ is positive, then $\sum_{k=1}^{\infty} u_k$ diverges.

The Limit Comparison Test says to make a ratio of the terms of two series and compute the limit. This test is most useful for series with rational expressions.

Example 8 Determine if $\sum_{k=1}^{\infty} \frac{k^4+6k^3-1}{7k^5+K^2}$ converges or diverges.

Solution

Just as with rational functions, the behavior of the series $\sum_{k=1}^{\infty} \frac{k^4+6k^3-1}{7k^5+K^2}$ when k goes to infinity behaves like the series with only the highest powers of k in the numerator and denominator: $\sum_{k=1}^{\infty} \frac{k^4}{7k^5}$. We will use the series $\sum_{k=1}^{\infty} \frac{k^4}{7k^5}$ to apply the Limit Comparison Test. First, when we simplify the series $\sum_{k=1}^{\infty} \frac{k^4}{7k^5}$, we get the series $\sum_{k=1}^{\infty} \frac{1}{7k}$. This is a harmonic series because $\sum_{k=1}^{\infty} \frac{1}{7k} = \frac{1}{7}\sum_{k=1}^{\infty} \frac{1}{k}$ and the multiplier $\frac{1}{7}$ does not affect the convergence or divergence. Thus, $\sum_{k=1}^{\infty} \frac{1}{7k}$ diverges. So, we will next check that the limit of the ratio of the terms of the two series is positive:

$$\lim_{k\to\infty} \frac{\frac{k^4+6k^3-1}{7k^5+k^2}}{\frac{1}{7k}} = \lim_{k\to\infty} \frac{7k^4+42k^3-7}{7k^4+K} = 1 > 0.$$

Using the Limit Comparison Test, because $\frac{1}{7k}$ diverges and the limit of the ratio is positive, then $\sum_{k=1}^{\infty} \frac{k^4+6k^3-1}{7k^5+K^2}$ diverges.

Unlike the Comparison Test, you do not have to compare the terms of both series. You may just make a ratio of the terms.

There is a **Simplified Limit Comparison Test**, which may be easier for you to use.

Theorem (The Simplified Limit Comparison Test)

Suppose $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ are series without negative terms. If $\lim_{k\to\infty} \frac{v_k}{u_k}$ is finite and positive, then either $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ both converge or $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ both diverge.

Example 9 Determine if $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ converges or diverges.

Solution

$\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ is a series without negative terms. To apply the Simplified Limit Comparison Test, we can compare $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ with the series $\sum_{k=1}^{\infty} \frac{2}{8^k}$, which is a convergent geometric series. Then $\lim_{k\to\infty} \frac{\frac{2}{8^k+5}}{\frac{2}{8^k}} = \lim_{k\to\infty} \frac{8^k}{8^k+5} = 1 > 0$. Thus, since $\sum_{k=1}^{\infty} \frac{2}{8^k}$ converges, then $\sum_{k=1}^{\infty} \frac{2}{8^k+5}$ also converges.

Review Questions

1. Write an example of a nondecreasing sequence.

2. Write an example of a sequence that is not nondecreasing.
3. Suppose $\{S_n\}$ is a nondecreasing sequence such that for each $M > 0$, there is an N, such that $S_n >$ M for all $n > N$. Does the sequence converge? Explain.
4. Determine if $\left\{\frac{5n^2}{2n^2+7}\right\}$ converges or diverges. If it converges, find its limit.
5. Determine if $\sum_{k=3}^{\infty}\left(\frac{1}{4}\right)^{k-1}$ converges or diverges. If it converges, find its sum.

Determine if each series converges or diverges.

6. $\sum_{k=1}^{\infty}\frac{1}{(4k+1)^{\frac{1}{2}}}$
7. $\sum_{k=1}^{\infty}\frac{2}{3k^5-4}$
8. $\sum_{k=1}^{\infty}\frac{5}{(k+1)(k+3)}$
9. $\sum_{k=1}^{\infty}\frac{7}{\sqrt[5]{k^2}}$
10. $\sum_{k=1}^{\infty}\frac{k^3+4k^2+1}{3k^6+2k^4}$
11. $\sum_{k=1}^{\infty}\frac{1}{(3k-1)^{\frac{5}{2}}}$
12. Maria uses the integral test to determine if $\sum_{k=1}^{\infty}\frac{3}{k^2}$ converges. She finds that $\int_{1}^{+\infty}\frac{3}{x^2} = 3$. She then states that $\sum_{k=1}^{\infty}\frac{3}{k^2}$ converges and the sum is 3. What error did she make?

Keywords

1. nondecreasing sequence
2. harmonic series
3. geometric series
4. $p-$series
5. Comparison Test
6. Integral Test
7. Limit Comparison Test
8. Simplified Limit Comparison Test

8.4 Series With Odd or Even Negative Terms

Learning Objectives

- Demonstrate an understanding of alternating series
- Apply the Alternating Series Test to an appropriate series
- Explain the difference between absolute and conditional convergence
- Determine absolute and/or conditional convergence of series

Alternating Series (harmonic, geometric, p-series)

Alternating series are series whose terms alternate between positive and negative signs. Generally, alternating series look like one of these expressions:

$u_1 - u_2 + u_3 - u_4 + \ldots$ or $-u_1 + u_2 - u_3 + u_4 - \ldots.$

Either the terms with the even indices can have the negative sign or the terms with the odd indices can have the negative sign. The actual numbers represented by the $u_i's$ are positive.

There are several types of alternating series. One type is the **alternating harmonic series:**

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

This series has terms that look like the harmonic series but the terms with even indices have a negative sign.

Another kind is the **alternating geometric series.** Here is one example:

$$\sum_{k=1}^{\infty} (-1)^k \left(\frac{2}{3}\right)^{k-1} = -1 + \frac{2}{3} - \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 - \ldots.$$

The odd-indexed terms of this series have the negative sign.

The **alternating** $p-$ **series** is another type of alternating series. An example could look like this:

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\sqrt[3]{k}} = 1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} - \ldots.$$

From all of these examples, we can see that the alternating signs depend on the expression in the power of -1 in the infinite series.

The Alternating Series Test

As its name implies, the **Alternating Series Test** is a test for convergence for series who have alternating signs in its terms.

Theorem (The Alternating Series Test)

The alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$ or $-u_1 + -u_1 + u_2 - u_3 + u_4 - \ldots$ converge if:

1. $u_1 \geq u_2 \geq u_3 \geq \ldots \geq u_k \geq \ldots$

and

2. $\lim_{k \to +\infty} u_k = 0$.

Take the terms of the series and drop their signs. Then the theorem tells us that the terms of the series must be nonincreasing and the limit of the terms is 0 in order for the test to work. Here is an example of how to use The Alternating Series Test.

Example 1

Determine if $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+5}{k^3+k}$ converges or diverges.

Solution

The series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+5}{k^3+k}$ is an alternating series. We must first check that the terms of the series are nonincreasing. Note that in order for $u_k \geq u_{k+1}$, then $1 \geq \frac{u_{k+1}}{u_k}$, or $\frac{u_{k+1}}{u_k} \leq 1$.

So we can check that the ratio of the $(k+1)$st term to the k th term is less than or equal to one.

$$\frac{u_{k+1}}{u_k} = \frac{\frac{(k+1)+5}{(k+1)^3+(k+1)}}{\frac{k+5}{k^3+k}} = \frac{(k+1)+5}{(k+1)^3+(k+1)} \times \frac{k^3+k}{k+5}$$

Expanding the last expression, we get:

$$\frac{u_{k+1}}{u_k} = \frac{(k+6)(k^3+k)}{(k^3+3k^2+4k+2)(k+5)} = \frac{k^4+6k^3+k^2+6k}{k^4+8k^3+19k^2+22k+10}.$$

Since k is positive and all the sum of the numerator are part of the denominator's sum, the numerator is less than the denominator and so, $\frac{u_{k+1}}{u_k} < 1$. Thus, $u_k \geq u_k + 1$ for all k. By the Alternating Series Test, the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+5}{k^3+k}$ converges.

Keep in mind that both conditions have to be satisfied for the test to prove convergence. However, if the limit condition is not satisfied, the infinite series diverges.

Alternating Series Remainder

We find the sequence of partial sums for an alternating series. A partial sum can be used to approximate the sum of the series. If the alternating series converges, we can actually find a bound on the difference between the partial sum and the actual sum. This difference, or remainder, is called the **error.**

Theorem (Alternating Series Remainder)

Suppose an alternating series satisfies the conditions of the Alternating Series Test and has the sum S. Let s_n be the nth partial sum of the series. Then $|S-s_n|\leq u_{n+1}$.

The main idea of the theorem is that the remainder $|S-s_n|$ cannot get larger than the $n+1$st term in the series, u_{n+1}. This is the term whose index is one more than the index of the partial sum used in the difference.

Example 2

Compute s_3 for the series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+5}{k^3+k}$ and determine the bound on the remainder.

Solution

First we compute the third partial sum to approximate the sum S of the series:

$$\begin{aligned} s_3 &= (-1)^2\frac{1+5}{1^3+1}+(-1)^3\frac{2+5}{2^3+2}+(-1)^4\frac{3+5}{3^3+3} \\ &= \frac{6}{4}-\frac{7}{10}+\frac{8}{30} \\ &= \frac{90-42+16}{60}=\frac{64}{60} \end{aligned}$$

The theorem tells us to use the next term in the series, u_4, to calculate the bound on the difference or remainder. Remember that the part $(-1)^{k+1}$ just gives the sign of the term and, so we just use the part $\frac{k+5}{k^3+k}$ to calculate u_4.

Thus $u_4=\frac{4+5}{4^3+4}=\frac{9}{68}$. Then $|S-s_n|=\left|S-\frac{64}{60}\right|<\frac{9}{68}\approx 0.13$. This tells us that the absolute value of the error or remainder is less than 0.13.

Absolute and Conditional Convergence

There are other types of convergence for infinite series: **absolute convergence** and **conditional convergence.**

Absolute Convergence

Let $\sum_{k=1}^{\infty}u_k=u_1+u_2+u_3\ldots+u_k+\ldots$ be an infinite series. Then the series is **absolutely convergent** if $\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\ldots+|u_k|+\ldots$ converges.

The infinite series $\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\ldots+|u_k|+\ldots$ is the series made by taking the absolute values of the terms of the series $\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\ldots+|u_k|+\ldots$.

The convergence of the series of absolute values tells us something about the convergence of the series.

Theorem

If $\sum_{k=1}^{\infty}|u_k|=|u_1|+|u_2|+|u_3|+\ldots+|u_k|+\ldots$ converges, then $\sum_{k=1}^{\infty}u_k=u_1+u_2+u_3+\ldots+u_k+\ldots$ also converges.

This tells us that if you can show absolute convergence, then the series converges.

If the series of absolute value diverges, we cannot conclude anything about the series.

Example 3

Determine if the series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3k^4-k}$ converges absolutely.

Solution

We find the series of absolute values: $\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{3k^4-k}\right|=\sum_{k=1}^{\infty}\frac{1}{3k^4-k}$, which behaves like the series $\sum_{k=1}^{\infty}\frac{1}{3k^4}$. This is a $p-$series with $p=4$. The series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{3k^4-k}$ converges absolutely and hence converges.

Example 4

Determine if the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ converges absolutely.

Solution

The series made up of the absolute values of the terms is $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2k+1} \right| = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$. This series behaves like $\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k}$, which diverges. The series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k+1}$ does not converge absolutely.

It is possible to have a series that is convergent, but not absolutely convergent. We know that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+1}$ is an alternating series. By the Alternating Series Tes we know that this series will converge if both of two subtests are satisfied, that is, if 1. $\mu \geq \mu_{k+1}$ for all $K\underline{\text{and}}$ 2. $\lim_{k\to\infty} \mu_k = 0$. In this case, $\mu = \frac{1}{2k+1}$ and $\mu_{k+1} = \frac{1}{2(K+1)+1} = \frac{1}{2k+3}$, so we know, since the second denominator is always larger than the first, that the first subtest $(\mu \geq \mu_{k+1})$ of the Alternating Series Test is satisfied. Then we can also see that $\lim_{k\to\infty} \frac{1}{2k+1} = 0$, and so the second subtest is also satisfied, and so the series converges.

Conditional Convergence

An infinite series that converges, but does not converge absolutely, is called a **conditionally convergent** series.

Example 5

Determine if $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges absolutely, converges conditionally, or diverges.

Solution

The series of absolute values is $\sum_{k=1}^{\infty} \frac{1}{k}$. This is the harmonic series, which does not converge. So, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ does not converge absolutely. The next step is to check the convergence $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. This will tell us if the series converges conditionally. Applying the Alternating Series Test:

The sequence $\frac{1}{1} > \frac{1}{2} > \frac{1}{3} > \ldots$ is nonincreasing and $\lim_{k\to\infty} \frac{1}{k} = 0$.

The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges. Hence, the series converges conditionally, but not absolutely.

Rearrangement

Making a **rearrangement** of terms of a series means writing all of the terms of a series in a different order. The following theorem explains how rearrangement affects convergence.

Theorem

If $\sum_{k=1}^{\infty} u_k$ is an absolutely convergent series, then the new series formed by a rearrangement of the terms of the series also converges absolutely.

This tells us that rearrangement does not affect absolute convergence.

Review Questions

Determine if the series converges or diverges.

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{3k^2+k}$
2. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2^k}$
3. Compute s_3 for $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4}{k^3}$.
4. The series $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2}$ converges according to the Alternating Series Test. Let $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2} = S$. Compute s_3 for $\sum_{k=1}^{\infty} (-1)^k \frac{5}{k^2}$ and determine the bound on $|s_3 - S|$.

5. The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ converges according to the Alternating Series Test. Let $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = S$. Compute s_4 for $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ and determine the bound on $|s_4 - S|$.

The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges according to the Alternating Series Test. Let $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = S$. Find the least value of n such that:

6. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.05$
7. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.005$
8. $\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} - S\right| < 0.0001$

Determine if each series converges absolutely, converges conditionally, or diverges.

9. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k}{2^k}$
10. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{2k^2+2}$
11. $\sum_{k=1}^{\infty} \frac{(-4)^{k+1}}{7k^2}$
12. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\frac{7}{2}}}$

Keywords

1. alternating series
2. alternating harmonic series
3. alternating geometric series
4. alternating $p-$series
5. Alternating Series Test
6. Alternating Series Remainder
7. conditional convergence
8. absolute convergence
9. rearrangement

8.5 Ratio Test, Root Test, and Summary of Tests

Ratio Test

We have seen the Integral Test,(Limit) Comparison Test and Alternating Series Test which impose conditions on the sign of a_n. We are going to introduce two tests for a stronger version of convergence that do not.

Definition

(Absolute convergence)

A series $\sum a_n$ is **absolutely convergent** if the series of the absolute values $\sum |a_n|$ is convergent.

To this end, we need to distinguish the other type of convergence.

Definition

(Conditional convergence)

A series $\sum a_n$ is **conditionally convergent** if the series is convergent but *not* absolutely convergent.

Theorem If $\sum a_n$ is absolutely convergent, then it is convergent.

The proof is quite straightforward and is left as an exercise. The converse of this theorem is not true. The series $\sum \frac{(-1)^{n-1}}{n}$ is convergent by the Alternating Series Test, but its absolute series, $\sum \frac{1}{n}$ (the harmonic series), is divergent.

Example 1 $\sum \frac{\cos\, n\theta}{n^2}$ is absolutely convergent since $\left|\frac{\cos\, n\theta}{n^2}\right| \le \frac{1}{n^2}$ for any $1 \le n, \theta$, and $\sum \frac{1}{n^2}$ is convergent (e.g. by the $p-$ test). Indeed, by the Integral and Comparison tests,$\sum \frac{\cos\, n\theta}{n^p}$ is absolutely convergent for any θ and $p > 1$.

The limit of the ratio $\left|\frac{a_n+1}{a_n}\right|$ gives us a comparison of the tail part (i.e $\sum_n$ Large a_n) of the series $\sum_{an}$ with a geometric series.

Theorem (The Ratio Test) Let $\sum_{an}$ be a series of non-zero numbers*.

(A) If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = \alpha < 1$, then the series is absolutely convergent.

(B) If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = \alpha > 1$ or $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = \infty$ then the series is absolutely divergent.

(C) If $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = \alpha = 1$ then the test is inconclusive.

$*$: we could ignore the zero-valued a_n's as far as the sum is concerned.

Proof. (A) The proof is by comparison with a geometric series.

If $\alpha < 1$, then $\alpha \; < \frac{\alpha+1}{2} \; < 1$. It follows from the definition of limit that there is an integer N, $\left|\frac{a_{n+1}}{a_n}\right| < \frac{\alpha+1}{2}$ for all $n \ge N$. Let $\beta = \frac{\alpha+1}{2}$ Then $|a_{N+1}| < \beta |a_N|, |a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|, \ldots$ and recursively we have $|a_{n+1}| < \beta^{n-N+1} |a_N|$ for $n \ge N$, and $\sum_{n=N}^{\infty} |a_n| < |a_N| \sum_{n=N}^{\infty} \beta^{n-N}$ which is finite. Combining with the finitely many terms, $\sum_{n=0}^{\infty} |a_n|$ is still finite.

(B) A similar argument concludes $\lim_{n\to\infty} a_n \ne 0$. So the series is divergent.

Example 2 Test the series $\sum \frac{A^n}{n!}$ for absolute convergence where A is a constant.

Solution. Let $a_n = \frac{A^n}{n!}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{A^{n+1}}{(n+1)!} \cdot \frac{n!}{A^n} = \frac{A}{n+1} \to \; 0$ as $n \to \infty$. So by the Ratio Test, the series is absolutely convergent for any constant A. Indeed, the sum is e^A which is very large for large A, but still finite.

We see the limitation of the Ratio Test is when $\lim \left|\frac{a_{n+1}}{a_n}\right|$ does not exist (not ∞) or is 1.

Example 3 (Ratio Test inconclusive) $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$, and $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_N}\right| = 1$ for both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. The former (harmonic series) diverges while the latter converges (by, say, the $p-$test).

Questions (related to the Ratio Test) What if

1. limits of $\left|\frac{a_{n+1}}{a_n}\right|$ exist separately for n odd and n even, i.e $\lim_{n\to\infty} \left|\frac{a_{2n+1}}{a_{2n}}\right|, \lim_{n\to\infty} \left|\frac{a_{2n}}{a_{2n-1}}\right|$ exist but are different?
2. $\lim_{n\to\infty} \left|\frac{a_{2n+1}}{a_{2n-1}}\right|, \lim_{n\to\infty} \left|\frac{a_{2n+2}}{a_{2n}}\right|$ exist but are different?

Exercises

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent with the Ratio Test and other tests if necessary:

1. $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$
2. $\sum_{n=1}^{\infty} e^{-2n} n!$
3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$
4. $\frac{2}{1.3} - \frac{3}{2.4} + \frac{4}{3.5} - \frac{5}{4.6} + \dots$

nth - root Test

If the general term a_n resembles an exponential expression, the following test is handy.

Theorem (The Root Test)

(A) If $\lim_{n\to\infty} |\sqrt[n]{a_n}| = \alpha < 1$, then the series is absolutely convergent.

(B) If $\lim_{n\to\infty} |\sqrt[n]{a_n}| = \alpha > 1$ or $\lim_{n\to\infty} |\sqrt[n]{a_n}| = \infty$, then the series is absolutely divergent.

(C) If $\lim_{n\to\infty} \left|\sqrt[n]{n}\right| = \alpha = 1$, then the test is inconclusive.

The proof is similar to that of the Ratio Test and is left as an exercise.

Example 1 Consider $\sum a_n = \sum \frac{1}{n^p}$ where $p > 0$. We already know it is convergent when $p > 1$. To apply the Root Test, we need $\lim_{n\to\infty} \sqrt[n]{n}$ which is 1 after some work. Alternatively, we could check $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ so $\lim_{n\to\infty} \sqrt[n]{n} = 1$ by the argument similar to the proof of the Ratio Test. The Root Test is also inconclusive.

Example 2 Test the series $\sum \left(\frac{n+1}{2n+3}\right)^n$ for convergence.

Solution. Let $a_n = \left(\frac{n+1}{2n+3}\right)^n$. Then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$. So the series is absolutely convergent.

What if we apply the Ratio Test?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+2}{2n+5}\right)^{n+1} / \left(\frac{n+1}{2n+3}\right)^n = \lim_{n\to\infty} \left(\frac{n+2}{2n+5}\right) \cdot \left[\frac{(n+2)(2n+3)}{(n+1)(2n+5)}\right]^n$$

We could still argue the limit is $\frac{1}{2}$ with some work. So we should learn to apply the right test.

Exercises

Determine whether the following series is absolutely convergent, conditionally convergent, or divergent with the Root Test and other tests if necessary:

1. $\sum_{n=1}^{\infty}\left(\frac{n}{n^2+1}\right)^n$
2. $\sum_{n=1}^{\infty}\left(\frac{(-1)^n(\ln\ n)}{n}\right)^n$
3. $\sum_{n=1}^{\infty}\left(\frac{n^n}{5^{3+2n}}\right)$
4. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \alpha$, then $\lim_{n\to\infty}|\sqrt[n]{a}_n| = \alpha$.
5. (Hard) Is the series $\sum_{n=1}^{\infty}\left(\frac{n^2-1}{n^2+1}\right)^n$ convergent?

Hint: $\left(\frac{n^2-1}{n^2+1}\right)^n = \sqrt[n]{\left[\left(\frac{n^2-1}{n^2+1}\right)^{n2}\right]}$ and $\lim_{h\to 0}(1+h)^{1/h}$ exists and equals e.

Summary of Procedures for Determining Convergence

We have seen various test for convergence of $\sum a_n$ in action. To summarize, the key phrase is " **recognize the form** of a_n".

It is sometimes difficult to choose the best convergence test for a particular series. Not all tests work on any given series, and even if a test works on a particular series, that test may still involve a lot of work in reaching a convergence conclusion. With experience, however, we can learn to apply the right test to a given series. At a minimum, we can learn which tests are easiest to apply, so that we can start with those easier tests if we have no other idea how proceed. The following is a summary of the various tests and when they might be useful:

TABLE 8.2:

	Test	Form of of a_n	Comments
1	No/little test	geometric, harmonic, $p-$test	clear answer
2	Test of divergence	$\lim_{n\to\infty} a_n \neq 0$	inexpensive test
3	Integral Test	corresp. integral in nice closed form	easy integration
4	Alternating Series	$\sum(-1)^n b_n$ (or $\sum(-1)^{n-1} b_n$)	check conditions on b_n
5	(Limit) Comparison	need companion known series	compare
6	Ratio Test*	recognize good form $\frac{a_{n+1}}{a_n}$	evaluate ratio
7	Root Test*	a_n resembles $c_n{}^n$	evaluate n^{th} root
8	Combination	composite of forms	combined methods

$*$: the inconclusive cases need other tests.

Example 1

$\sum\frac{1}{n(\ln\ n)}$ diverges by the Integral Test since $\int\frac{1}{x(\ln\ x)}dx = \ln|\ln|x|| + C$ diverges.

Example 2 $\sum\sin\ \frac{1}{n}$ diverges by Limit Comparison Test (against the harmonic series) since $\lim_{n\to\infty}\frac{\sin\ \frac{1}{n}}{\frac{1}{n}} = \lim_{h\to 0}\frac{\sin\ h}{h} = 1$ and the harmonic series diverges.

Example 3 $\sum\frac{1}{n}\sin\left(\frac{1}{n}\right)$ converges with Limit Comparison Test (against $\sum\frac{1}{n^2}$) since $\lim_{n\to\infty}\frac{\frac{1}{n}\sin\ \frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\sin\ \frac{1}{n}}{\frac{1}{n}} = \lim_{h\to 0}\frac{\sin\ h}{h} = 1$ and the latter series converges.

Example 4 Convergence of $\sum (c_n)^n$ is determined with $\lim_{n\to\infty} |c_n|$ by the Root Test. For example $\sum(\sqrt[n]{3}-\sqrt[n]{2})^n$, converges since $\lim_{n\to\infty}(\sqrt[n]{3}-\sqrt[n]{2}) = 1-1 = 0$, but $\sum(\sqrt[n]{2}-1)$ (no exponent n) diverges by rationalizing the numerator:

$\sqrt[n]{2}-1 = \frac{2-1}{2^{\frac{n-1}{n}}+2^{\frac{n-2}{n}}+\ldots+1} \geq \frac{2-1}{n.2^{\frac{n-1}{n}}} = \frac{1}{2}\frac{\sqrt[n]{2}}{n}$ and applying Limit Comparison test with $\sum \frac{1}{2n}$.

The Root Test is inconclusive on $\sum(1+\frac{1}{n})^n$, but the simpler Test for Divergence confirms its divergence since $(1+\frac{1}{n})^n > 1$ always.

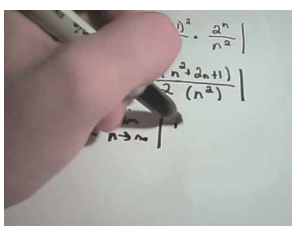
Example 5 $\sum \frac{(-1)^{n-1}}{n^q}$ is convergent for $q > 0$ by the Alternating Series Test since $\lim_{n\to\infty} \frac{1}{n^q} = 0$ and $\frac{1}{n^q} \geq \frac{1}{(n+1)^q}$. It is absolutely convergent for $q > 1$ by the $p-$test. So it is conditionally convergent for $0 < q \leq 1$.

Example 6 $\sum_{n=1}^{\infty} \frac{n.2^n}{n!} = \sum_{n=2}^{\infty} \frac{2.2^{n-1}}{(n-1)!} = 2\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is absolutely convergent by the Ratio Test since $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0$.

Example 7 Consider the series $\sum \frac{\sqrt{n+1}}{n^2-10n+1}$. Notice $n^2-10n+1$ is never 0 and is positive for $n \geq 10$, we could ignore the terms before $n = 10$. Dropping the lower powers of n leads to the candidate $\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{\frac{3}{2}}}$ for applying Limit Comparison Test since $\lim_{n\to\infty} \frac{\sqrt{n+1}}{n^2-10n+1} \cdot \frac{n^2}{\sqrt{n}} = 1$. So the series is (absolutely) convergent by the $p-$test. A combination of tests is applied.

Multimedia Links

For video presentations of the varying tests for convergence **(23.0)**, see Just Math Tutoring, Using the Ratio Test to Determine if a Series Converges (7:38)



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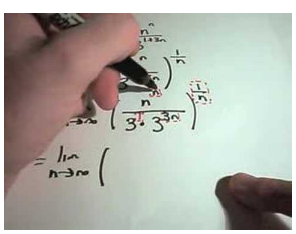
; Just Math Tutoring, Root Tests for Series (10:07) 4



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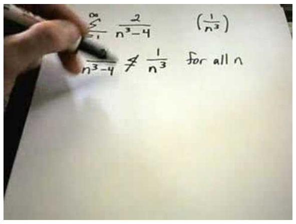
; Just Math Tutoring, Limit Comparison Test and Direct Comparison Test (7:35)



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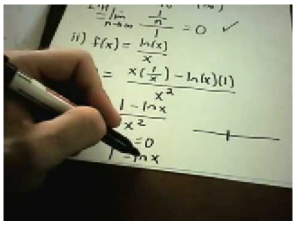
; Just Math Tutoring, Alternate Series Test (7:18).



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Exercises

1. For what values of p is the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln\ n)^p}$ convergent?

Test the following series for convergence or divergence:

2. $\sum_{n=1}^{\infty} \sin^2 \frac{1}{n}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

4. $\sum_{n=1}^{\infty} (\sqrt[n]{5} - \sqrt[n]{3})^n$

5. $\sum_{n=1}^{\infty} (\sqrt[n]{5} - \sqrt[n]{3})$

6. For what values of c is the series $\sum_{n=1}^{\infty} \frac{3^n}{2^n+c^n}$ convergent?

8.6 Power Series

Power Series and Convergence

Definition

(Power Series)

A **Power Series** is a series of the form

(PS1) $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

where x is a variable and the a_n's are constants (in our case, real numbers) called the **coefficients** of the series.

The summation sign $\sum$ is a compact and convenient shorthand notation. Readers unfamiliar with the notation might want to write out the detail a few times to get used to it.

Power series are a generalization of polynomials, potentially with infinitely many terms. As observed, the indices n of a_n are non-negative, so no negative integral exponents of x, e.g. $\frac{1}{x}$ appear in a power series.

More generally, a series of the form

(PS2) $\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$

is called a power series in $(x - x_0)(1 = (x - x_0)^0)$ or a power series **centered** at x_0 ((PS1) represents a series centered at $x = 0$).

Given any value of x, a power series ((PS1) and (PS2)) is a series of numbers. The first question is:

Is the power series (as in (PS1) or (PS2)) a function of x?

Since the series is always defined at $x = 0$ (resp. $x = x_0$), the question becomes:

For what value of x is a power series convergent?

The answers are known for some series. Convergence tests could be applied on some others.

Example 1 Let $r \neq 0$ and x_0 be real.

$\sum_{n=0}^{\infty} (r(x - x_0))^n = 1 + r(x - x_0) + r^2(x - x_0)^2 + r^3(x - x_0)^3 + \dots$ is absolutely convergent and equals $\frac{1}{1 - r(x - x_0)}$ for $|r(x - x_0)| < 1$, i.e. $|x - x_0| < \left|\frac{1}{r}\right|$, and diverges otherwise.

Let $r = 1; x_0 = 0$. Then $\sum x^n$ is the power series for $\frac{1}{1-x}$ on $(-1, 1)$. Let $r = -1; x_0 = 2$. Then $\sum (-1)^n (x - 2)^n$ is the power series for $\frac{1}{1-(-1)(x-2)} = \frac{1}{x-1}$ on $(1, 3)$. So $\sum x^n$ and $\sum (-1)^{n-1} (x - 2)^n$ are the power series for the same function $\frac{1}{1-x}$ but on different intervals. There is a more detailed discussion in §8.7.

Example 2 $\sum_{n=0}^{\infty} x^{2^n}$ is absolutely convergent for $|x| < 1$ by Comparison Test (against $\sum_{n=0}^{\infty} x^n$) and diverges for $|x| \geq 1$ by the Test for Divergence.

Exercises

1. Write a power series $\sum_{n=0}^{\infty} a_n (x + 2)^n$ centered at $x = -2$ for the same function $\frac{1}{1-x}$ in Example 6.1.1. On what interval does equality hold?

Hint: Substitute $y = x + 2$ in $\frac{1}{1-x}$.

2. Discuss the convergence of the series $\sum_{n=0}^{\infty} 2^n x^{2^n}$.

Hint: Apply a combination of tests in §8.5.

Interval and Radius of Convergence

The following theorem characterizes the values of x where a power series is convergent.

Theorem (Interval of convergence)

Given a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Exactly one of the following three describes all the values where the series is convergent:

(A) The series converges exactly at $x = x_0$ only.

(B) The series converges for all x.

(C) There is a real number $R_c > 0$ that the series converges if $|x-x_0| < R_c$ and diverges if $|x-x_0| > R_c$.

This R_c is unique for a power series, called the **radius of convergence**. By convention $R_c = 0$ for case (A) and $R_c = \infty$ for case (B). The only two values of x the Theorem cannot confirm are the *endpoints* $x = x_0 \pm R_c$. In any case, the values x where the series converges is an interval, called the **interval of convergence**. It is the singleton x_0 for case (A) and $(-\infty,\infty)$ for case (B). For case (C), it is one of the four possible intervals: $(x_0-R_c, x_0+R_c), (x_0-R_c, x_0+R_c], [x_0-R_c, x_0+R_c)$, and $[x_0-R_c, x_0+R_c]$. Here, the endpoints must be checked separately for convergence.

Example 1 Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$.

Solution. Let $b_n = \frac{x^n}{n^2}$. Then $\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = \left|\left(\frac{1}{1+\frac{1}{n}}\right)^2 x\right| \to |x|$ as $n \to \infty$. So the series is absolutely convergent for $|x| < 1$ (and divergent for $|x| > 1$) by the Ratio Test. $R_c = 1$. This leaves the endpoint values to check.

If $x = \pm 1$, then the series is absolutely convergent by the $p-$test. Hence the series is absolutely convergent for $|x| \leq 1$. The interval of convergence is $[-1,1]$.

Example 2 If the series $\sum_{n=0}^{\infty} a_n 2^n$ converges, then $\sum_{n=0}^{\infty} a_n x^n$ is convergent at $x = 2$, i.e. 2 is inside the interval of convergence. So $R_c \geq 2$. Conversely, if the series $\sum_{n=0}^{\infty} a_n(-3)^n$ diverges, then $\sum_{n=0}^{\infty} a_n x^n$ is divergent at $x = -3$, i.e. -3 is outside the interval of convergence. So $R_c \leq |-3| = 3$.

Exercises

Find the radius of convergence and interval of convergence of the following series.

1. $\sum_{n=1}^{\infty} nx^n$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$
3. $\sum_{n=1}^{\infty} \frac{x^{n/3}}{n!}$
4. $\sum_{n=1}^{\infty} \sqrt{n}(x-x_0)^n$
5. Given $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 5$ and diverges at $x = -7$. Deduce where possible, the convergence or divergence of these series:

    a. $\sum_{n=0}^{\infty} a_n$
    b. $\sum_{n=0}^{\infty} a_n 3^n$
    c. $\sum_{n=0}^{\infty} a_n(-8)^n$
    d. $\sum_{n=0}^{\infty} a_n(9)^n$
    e. $\sum_{n=0}^{\infty} a_n(6)^n$

Term-by-Term Differentiation of Power Series

The goal of the next 3 sections is to find power series representations of certain classes of functions, namely derivatives, integrals and products.

In the study of differentiation (resp. integration), we have found the derivatives (resp. integrals) of better known functions, many with known power series representations. The power series representations of the derivatives (resp. integrals) can be found by term-by-term differentiation (resp. integration) by the following theorem.

Theorem (Term-by-term differentiation and Integration)

Suppose $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has radius of convergence R_c. Then the function f defined by $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is differentiable on (x_0-R_c, x_0+R_c) and

(A) $f'(x)=\sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$,

(B) $\int f(x)dx=\sum_{n=0}^{\infty} \frac{a_n(x-x_0)^{n+1}}{n+1}+C$, and these power series have same radius of convergence R_c.

(A) means (dropping x_0) *the derivative of a power series is the same as the term-by-term differentiation of the power series*:

$\frac{d}{dx}\sum_{n=0}^{\infty} a_nx^n=\sum_{n=0}^{\infty}\frac{d}{dx}(a_nx^n)$ and

(B) means *the integral of a power series is the same as the term-by-term integration of the power series*:

$$\int\sum_{n=0}^{\infty}(a_nx^n)dx=\sum_{n=0}^{\infty}\int(a_nx^n)dx.$$

Example 1 Find a power series for $g(x)=\frac{1}{(1-x)^2}$ and its radius of convergence.

Solution. We recognize $g(x)$ as the derivative of $\frac{1}{1-x}$ whose power series representation is $\sum_{n=0}^{\infty} x^n$ with radius of convergence $R_c=1$. By (A), $g(x)=\frac{d}{dx}\sum_{n=0}^{\infty} x^n=\sum_{n=1}^{\infty} nx^{n-1}$ and has radius of convergence 1.

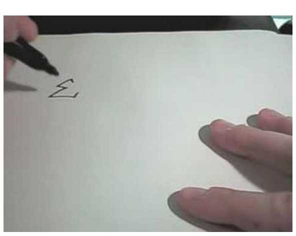
Exercises

Find a power series and the radius of convergence for the following functions:

1. $\frac{x}{2-x}$
2. $\frac{x^2}{(1-x)^3}$
3. $\frac{2x}{(1-x)^3}+\frac{3x^2}{(1-x)^4}$

Multimedia Links

For a video presentation of finding the interval of convergence of a power series **(24.0)**, see Just Math Tutoring, Power Series, finding the interval of convergence (9:47).



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Term-by-Term Integration of Power Series

Example 1 Find a power series for $h(x) = \tan^{-1} x$ and its radius of convergence.

Solution. We recognize $h(x)$ as the antiderivative of $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$.

By Term-by-Term Theorem (B), $h(x) = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + C$ and has radius of convergence 1.

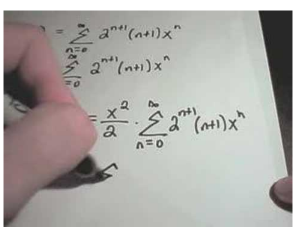
Then $C = \tan^{-1} 0 = 0$ and $h(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

Exercises

1. Find a power series for $\ln(1+x^2)$ and find the radius of convergence.
2. Express $\int \tan^{-1} x dx$ as a power series and find the radius of convergence.
3. Find a power series for $\ln(1+x+x^2)$ in x and in $x+\frac{1}{2}$, and find the radius of convergence.

Multimedia Links

For a video presentation showing how to differentiate and integrate power series **(25.0)**, see Just Math Tutoring, Differentiating and Integrating a Power Series (10:10).



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Series Multiplication of Power Series

Definition

(Series Multiplication) The **power series product** of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ is a power series $\sum_{n=0}^{\infty} c_n x^n$ defined by $c_n = \sum_{i=0}^{n} a_i b_{n-i} = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$ (like polynomials).

A result is: *the product of power series is the power series of the product.*

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on a common interval $|x| < R_a b$, then their product power series $\sum_{n=0}^{\infty} c_n x^n$ also converges on $R_a b$ and is the power series for the product function $f(x)g(x)$.

Example 1 Find a power series for $\frac{1}{(1-x)(1-2x)}$.

Solution. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with radius of convergence 1 and $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$ with radius of convergence $\frac{1}{2}$.

So for $|x| < \frac{1}{2}$,

$$\begin{aligned}\frac{1}{(1-x)(1-2x)} &= (1+x+x^2+\ldots)(1+2x+4x^2+\ldots)\\ &= 1+3x+7x^2+15x^3+\ldots \text{by the above result}\\ &= \sum_{n=0}^{\infty}(2^{n+1}-1)x^n.\end{aligned}$$

Exercises

1. Find the first 4 terms of a power series for $\frac{1}{\sqrt{(1+x)(1+2x)}}$.
2. Find a power series for $\frac{1}{(1-rx)(1-sx)}$ where r, $s > 0$ are real numbers and
   a. $r \neq s$
   b. $r = s$.

8.7 Taylor and Maclaurin Series

Taylor and Maclaurin Polynomials

We know the linear approximation function $L_1(x)$ to a (smooth) function $f(x)$ at $x = x_0$ is given by the tangent line at the point. $L_1(x_0) = f(x_0)$ and $L_1'(x_0) = f'(x_0)$. Indeed, this is the only linear function with these 2 properties.

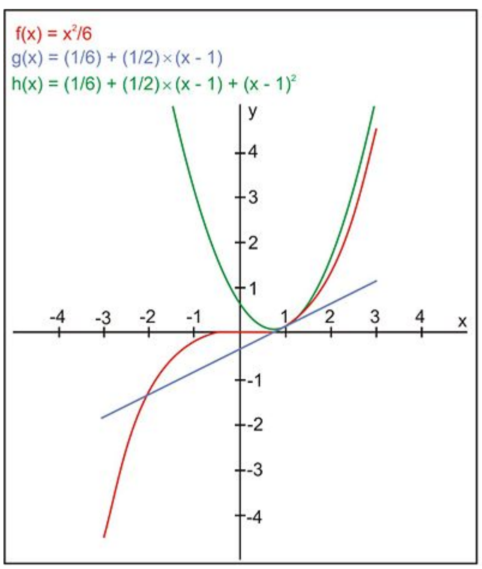
Theorem ($n^{th}-$degree Taylor polynomial) Given a function f with continuous n^{th} derivative in an open interval containing $x*_0$. There exists unique $n_{th}-$degree polynomial $p(x)$ with $p^{(j)}(x_0) = f^{(j)}(x_0)$, for $0 \leq j \leq n$.

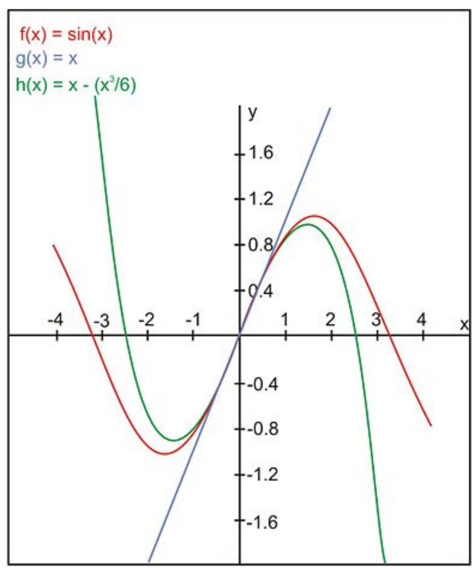
$\star$: the functions in this text have continuous derivatives at the center x_0 unless otherwise stated.

This polynomial

$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x-0)}{n!}(x - x_0)^n)$ is called the $n^{th}-$degree Taylor polynomial of f at x_0. If $x_0 = 0$, it is called the $n^{th}-$ degree Maclaurin polynomial of f and denoted by $M_n(x)$. $R_n(x) = f(x) - T_n(x)$ is the remainder of the Taylor polynomial.

Example 1 Let $f(x) = \frac{x^3}{6}, x_0 = 1$. Then $f'(x) = \frac{x^2}{2}$ and $f(x) = x$. So $f'(1) = \frac{1}{2}$ and $f(1) = 1$. Hence $T_1(x) = \frac{1}{6} + \frac{1}{2}(x - 1), T_2(x) = \frac{1}{6} + \frac{1}{2}(x - 1) + (x - 1)^2$, and $T_3(x) = f(x)$ itself.





Example 2 Let $f(x) = \sin x, x_0 = 0$ and take $n = 3$. Then $f(x) = \cos x$, $f''(x) = -\sin\ x, f'''(x) = -\cos\ x$. So $f'(0) = 1, f''(0) = 0, f'''(0) = -1$. $M_3(x) = x - \frac{1}{3!}x^3 = x - \frac{1}{6}x^3$ is the third-degree Maclaurin polynomial of f.

Example 3 Find the second-degree Taylor polynomial of $f(x) = \tan x$ at $x_0 = \frac{\pi}{4}$. Solution. $f'(x) = \sec^2\ x$ and $f''(x) = 2\ \sec\ x \cdot \sec\ x\ \tan\ x = 2\ \sec^2\ x\ \tan\ x$. So $f'\left(\frac{\pi}{4}\right) = 2$ and $f''\left(\frac{\pi}{4}\right) = 2 \cdot 2 = 4$. Then $T_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$.

Exercises Find the Taylor series of the following functions at the given x_0 with given degree n.

1. $f(x) = e^x$ at $x = 0, n = 3$

2. $f(x) = \ln\ x$ at $x = 1, n = 4$
3. $f(x) = 1 + x + x^2 + x^3 + x^4$ at $x = -1, n = 4$

Taylor and Maclaurin Series

Definition

(Taylor Series of f)

The Taylor series of a function f at $x = x_0$ is the power series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

taking all the terms of the Taylor polynomials. The Maclaurin series $M(x)$ of f is the Taylor series at $x = 0$.

Example 1 Find the Maclaurin series of $f(x) = \cos x$.

Solution. $f'(x) = -\sin\ x,\ f'(x) = -\cos\ x,\ f''(x) = \sin\ x,\ f'''(x) = \cos\ x,\ f^{(4)}(x) = -\sin\ x,\ f^{(5)}(x) = -\cos\ x, \dots$.

Notice the pattern repeats every 4 terms. So $f'(0) = 0, f'(0) = -1, f''(0) = 0, f'''(0) = 1, f^{(4)}(0) = 0, x, f^{(5)}(0) = -1 \dots$.

The Maclaurin series of $f(x) = \cos x$ is

$$M(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots = \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

Exercises Find the Taylor series of following functions at the given x_0.

1. $f(x) = \frac{1}{1-x}$ at $x = 0$
2. $f(x) = e^x$ at $x = 1$
3. $f(x) = \frac{1}{x}$ at $x = 2$

Convergence of Taylor and Maclaurin Series

Since $f^{(n)}(x)$ is defined for all functions f in this text, the Taylor series $T(x)$ of f is always defined. As for power series in general, the first question is:

Is $T(x)$ convergent at $x = a$? There is no guarantee except at $a = x_0$. The second question is:

If $T(x)$ converges at $x = a$, does it equal $f(a)$? The answer is negative as show by the function:

$$f_1(x) = \begin{cases} e^{-\frac{1}{x^2}}\ x \neq 0 \\ 0\ x = 0 \end{cases}$$

Then

$$f_1'(x) = \begin{cases} \left(\frac{2}{x^3}\right) \cdot e^{-\frac{1}{x^2}} \; x \neq 0 \\ 0 \; x = 0 \end{cases}$$

It can be verified that $f_1'(0) = 0, f_1''(0) = 0, f_1'''(0) = 0, \ldots$.

So the Maclaurin series is 0, clearly difierent from f except at $x = 0$.

Nevertheless, here is a positive result.

Theorem If f has a power series representation at $x = x_0$, i.e.

$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $|x - x_0| < R_c$, then the coefficients are given by $a_n = \frac{f^{(n)}(x_0)}{n!}$.

So any power series representation at $x = x_0$ has the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Exercises

1. Find the higher order derivatives of the function $f_1(x)$ above thus recursively showing $f_1^{(n)}(0) = 0$ for $n \geq 0$
2. Verify the Theorem using term-by-term difierentiation.

Taylors Formula with Remainder, Remainder Estimation, Truncation Error

Recall the remainder $R_n(x)$ of the $n^{th}-$ degree Taylor polynomial at $x = x_0$ is given by $R_n(x) = f(x) - T_n(x)$.

Theorem (Convergence of Taylor series)

If $\lim_{n\to\infty} R_n(x) = 0$ for $|x - x_0| < R_c$, then f is equal to its Taylor Series on the interval $|x - x_0| < R_c$.

The above condition $\lim_{n\to\infty} R_n(x) = 0$ could be achieved through the following bound.

Theorem (Remainder Estimation)

If $|f^{(n+1)}(x)| \leq M$ for $|x - x_0| \leq r$, then we have the following bound for $R_n(x)$:

$|R_n(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1}$ for $|x - x_0| \leq r$.

Example 1 The function e^x is equal to its Maclaurin series for all x. Proof. Let $f(x) = e^x$. We need to find the above bound on $R_n(x)$.

If $|x| \leq r, f^{(n)}(x) = e^x \leq e^r$ for $n \geq 0$ and the remainder estimation gives $|R_n(x)| \leq \frac{e^r}{(n+1)!} |x|^{n+1}$ for $|x| \leq r$.

Since $\lim_{n\to\infty} \frac{e^r}{(n+1)!} |x|^{n+1} = e^r \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0, \lim_{n\to\infty} |R_n(x)| = 0$ by the squeeze Theorem. So $\lim_{n\to\infty} R_n(x) = 0$. Hence e^x is equal to its Maclaurin series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Example 2 (Truncation Error) What is the truncation error of approximating $f(x) = \sqrt{1+x}$ by its third-degree Maclaurin polynomial in for $|x| \leq 0.1$.

Solution.

$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}, f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}} = -\frac{1}{4}(1+x)^{-\frac{3}{2}},$$
$$f'''(x) = \left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} = \frac{3}{8}(1+x)^{-\frac{5}{2}}, f^{(4)}(x) = \left(\frac{3}{8}\right)\left(-\frac{5}{2}\right)(1+x)^{-\frac{7}{2}} = \frac{15}{16}(1+x)^{-\frac{7}{2}}.$$

For $|x| \leq 0.1, |f^{(4)}(-0.1)| = \frac{15}{16}\frac{1}{(1+(-0.1))^{\frac{7}{2}}} \leq 1.5.$

So $|R_3(x)| \leq (1.5)\cdot\frac{1}{4!}|(-0.1)|^4 \leq \frac{1.5}{24}(0.1)^4 \approx 6.25\times 10^{-6}$. This is the truncation error of approximating by the third-degree Maclaurin polynomial.

The following applet illustrates approximating functions with Taylor Series. You can change the center of the series and also observe how the error changes for the estimation at a particular value of x where the error is $f(x) - T_n(x)$. Taylor Series and Polynomials Applet .

Exercises

1. Find the power series representation of $f(x) = \sin x$ at $x = 0$ for all x. Why is it the Maclaurin series?
2. Find the power series representation of $f(x) = \cos x$ at $x = \frac{\pi}{3}$ for all x. Why is it the Taylor series at $x = \frac{\pi}{3}$?
3. What is the truncation error of approximating $f(x) = \sqrt{1+x}$ by its fourth-degree Maclaurin series in for $|x| \leq 0.1$.

Combining Series, Eulers Formula

In many cases, we could find Taylor (Maclaurin) series of functions from existing series without going through the proof that $\lim_{n\to\infty} R_n(x) = 0$ Examples are products, quotients and some sine and cosine functions.

Example 1 Find the Maclaurin series of $f(x) = x\sin x$ for all x.

Solution. The Maclaurin series of $\sin x$ is $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$. So the Maclaurin series of $x\sin x$ is $x\sin x = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+1)!}$

Example 2 Find the Maclaurin series of $f(x) = \cos^2 x$ for all x.

Solution. We could avoid multiplying the Maclaurin series of $\cos x$ with itself, by applying: $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ on the Maclaurin series of $\cos 2x = \sum_{n=0}^{\infty}\frac{(-1)^n (2x)^{2n}}{(2n)!}$, giving

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n (2x)^{2n}}{2\cdot(2n)!}$$

For any real number θ, $e^i\theta = \cos + i\sin\theta$ where $i = \sqrt{-1}$ is the imaginary unit. This is the **Euler's Formula**.

Euler's formula combines the complementary *sine* and *cosine* functions into the simpler exponential function and heavily applies the separation of real and imaginary parts of complex numbers.

Example 3 Find the Maclaurin series of $\cos x$ and $\sin x$ for all x through e^{ix}.

Solution. $\cos x + i\sin x = e^{ix}$ which has a Maclaurin series

$$e^{ix} = \sum_{n=0}^{\infty}\frac{(ix)^n}{n!} = \sum_{m=0}^{\infty}\frac{(ix)^{2m}}{(2m)!} + \sum_{m=0}^{\infty}\frac{(ix)^{2m+1}}{(2m+1)!}$$

by dividing into sum of odd and even indices. So $\cos\ x + i\sin\ x = \sum_{m=0}^{\infty}(-1)^m \frac{(x)^{2m}}{(2m)!} + i\sum_{m=0}^{\infty}(-1)^m \frac{(x)^{2m+1}}{(2m+1)!}$

The Maclaurin series of $\cos x$ and $\sin x$ follow by separately taking the real and imaginary parts.

Exercises

1. Find and compare the Maclaurin series for $\sin x \cos x$ and $\sin 2x$.

2. Find the Maclaurin series of $\frac{x}{e^x}$ for all x.

Hint: would you divide x by e^x ?

3. Find the Maclaurin series of $\cos 3x$ and $\sin 3x$ for all x using Euler's formula.

4. Find expressions for the series $\sum_{n=0}^{\infty} \cos\ n\theta\ x^n$ and $\sum_{n=0}^{\infty} \sin\ n\theta\ x^n$ for all θ and $|x| < 1$ using Euler's formula.

Binomial Series

We have learned the Binomial Theorem for positive integer exponents:

$$\begin{aligned}(a+b)^n &= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n (BE)\\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k\end{aligned}$$

where the Binomial coefficients are denoted by

$\binom{n}{0} = 1$ and $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$ for $k \geq 1$

AS a simple Binomial function, take $a = 1$ and $b = x$. Then $(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k$

Let r be a real number and $f(x) = (1+x)^r$. Is $f(x)$ equal to a series in the form of (BE) except that there may be an infinite series? The answer is yes.

Theorem (Binomial Series) Let r be a real number and $|x| < 1$. Then

$$\begin{aligned}(1+x)^r &= 1 + \text{rx} + \frac{r(r-1)}{k!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3\\ &= \sum_{k=0}^{n}\binom{r}{k}x^k\end{aligned}$$

where the Binomial coefficients are denoted by

$\binom{n}{0} = 1$ and $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$ for $k \geq 1$

where $\binom{r}{0} = 1$ and $\binom{r}{k} = \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k \geq 1$

Example 1 Find a power series representation of $\sqrt{1+x}$.

Solution. We need to compute the Binomial coefficients for $r = -\frac{1}{2}$

$$\begin{aligned}\binom{\frac{1}{2}}{k} &= \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2k-3}{2}\right)}{k!} = \frac{(-1)^{k-1}1\cdot 3\cdot 5\ldots(2k-3)}{2^k k!}\\ &= \frac{(-1)^{k-1}(2k-2)!}{2.4.6\ldots(2k-2))2^k k!} = \frac{(-1)^{k-1}(2k-2)!}{2^{k-1}k!(k-1)2^k k!} = \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!}\end{aligned}$$

So if $|x|< 1$, $\sqrt{1+x} = \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!} x^k$

Example 2 Find a power series representation of $\frac{1}{(1-x)^m}$ where m is a positive integer.

Solution. We need to compute the Binomial coefficients for $r = -m$ (and will replace x by $-x$)

$$\begin{aligned}
\binom{-m}{k} &= \frac{(-m)(-m-1)(-m-2)\dots(-m-k+1)}{k!} \\
&= \frac{(-1)^k m(m+1)(m+2)\dots(m+k-1)}{k!} = (-1)^k \binom{m+k-1}{k} \\
\text{So } \frac{1}{(1-x)^m} &= \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} (-x)^k \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} (-1)^k x^k \\
&= \sum_{k=0}^{\infty} (-1)^{2k} \binom{m+k-1}{k} x^k \\
&= \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k
\end{aligned}$$

Exercises

1. Find a power series representation of $\frac{1}{\sqrt{1+x}}$ at $x = 0$.
2. Find a power series representation of $\frac{1}{(2-x)^2}$ at $x = 0$.
3. Notice $1+x+x^2 = x + \frac{3}{4} + \left(x+\frac{1}{2}\right)^2$. Find a power series representation of $\sqrt{(1+x+x^2)}$ at $x = -\frac{1}{2}$. In what interval is the equality true?

Choosing Centers

Taylor Series (indeed Taylor polynomials of lower degrees) often provide good approximation of functions. However, the choice of center could determine

1. whether the intended value of x is inside the interval of convergence
2. rate of convergence, i.e. how many terms to take to achieve prescribed degree of accuracy

For frequently used functions, the first choice may be the standard center (see the list at the end of this section).

Example 1 Approximate ln $0:99$

Solution. Since .99 is close to the center $x = 1$, we use the standard Taylor series for $\ln(1-x)$.

$$\ln\ 0.99 = \ln(1-0.01) \approx -0.01 - \frac{(0.01)^2}{2} \approx -0.01005$$

Then we may be able to deduce a useful Taylor Series centered close to the given x.

Example 2 Approximate $\sin(1.1)$ to 4 decimal places.

Since 1.1 is close to $\frac{\pi}{3}$, we would try to find a Taylor Series of $\sin x$ at $x_0 = \frac{\pi}{3}$. Let $f(x) = \sin x$. Then

$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x$ and $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ f'\left(\frac{\pi}{3}\right) = \frac{1}{2},\ f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

This pattern repeats and $\lim_{n\to\infty} R_n(x) = 0$ can be checked as in the case $x_0 = 0$. So the Taylor Series is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

Hence

$$\begin{aligned}\sin(1.1) &\approx \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}(0.0528)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}(0.0528)^{2n+1}\\ &\approx 0.86481823 + 0.02638773 \text{ taking 2 terms from each sum}\\ &\approx 0.8912\end{aligned}$$

We may also apply algebraic manipulation to standard Taylor Series.

Example 3 Approximate $\frac{1}{1.9^2}$ to 4 decimal places.

Solution. There is standard Taylor Series:

$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} nx^{n-1}$ for $|x| < 1$ through term-by-term differentiation of the series for $\frac{1}{1-x}$ ($\frac{1}{(1-x)^2}$ is inadequate at $x = -0.9$).

Since 1.9 is close to 2, we consider

$\frac{1}{(2-x)^2} = \frac{1}{4}\frac{1}{(1-\frac{x}{2})^2} = \frac{1}{4}\sum_{k=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1}$ for $|x| < 2$.

So we take $x = 0.1$ and then

$$\frac{1}{1.9^2} = \frac{1}{4}\sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1} \approx \frac{1}{4}(1+2(0.05)+3(0.05)^2+4(0.05)^3) \approx 0.2770.$$

Exercises

1. Approximate $\ln\, 0.9$ to 4 decimal places.

2. Approximate $\sin(0.8)$ to 6 decimal places.

Hint: consider center $\frac{\pi}{4}$

3. Approximate $\frac{1}{9^3}$ to 6 decimal places.

Evaluating Nonelementary Integrals

There are many simple-looking functions that have no explicit formula for their integral in the form of elementary functions. We could write their in-tegrals as Taylor Series in their interval of convergence.

Example 1 Find a power series representation of $\int \frac{e^x}{x}dx$

Solution. Since $\frac{e^x}{x}$ is not defined at $x = 0$, we apply the Taylor Series of e^x at, say, $x = 1$ by writing $e^x = e \cdot e^{x-1}$ with a change of variable $u = x - 1$.

$$\frac{e^x}{x} = \frac{e \cdot e^u}{1+u} = e \cdot \frac{1}{1+u} \sum_{n=0}^{\infty} \frac{(u)^n}{n!}$$
$$= e\left(1 - u + u^2 - u^3 + \ldots\right)\left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{2!} + \ldots\right)$$
$$= e\left(1 + \frac{1}{2}u^2 - \frac{1}{3}u^3 + \frac{3}{8}u^4 - \frac{11}{30}u^5 + \ldots\right) \text{ for } |u| < 1$$

so $\int \frac{e^x}{x} dx = e\left(u + \frac{1}{6}u^3 - \frac{1}{12}u^4 + \frac{3}{40}u^5 - \frac{11}{180}u^6 + \ldots\right)$ where $u = x - 1$

Example 2 Find the power series representation of $\int \frac{\sin x^2}{x} dx$

Solution. Direct substitution of x^2 in the Maclaurin Series of $\sin x$ gives $\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1}$ and $\frac{\sin x^2}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+1}$

So

$$\int \frac{\sin x^2}{x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{4n+1} dx$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+2}}{4n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)} \frac{x^{4n+2}}{(2n+1)!}$$

Exercises

1. Find the power series representation (Maclaurin Series) of $\int e^{x^2} dx$ and approximate $\int_0^1 e^{x^2} dx$ to 6 decimal places.
2. Find the power series representation (Maclaurin Series) of $\int \sin x^2 dx$.

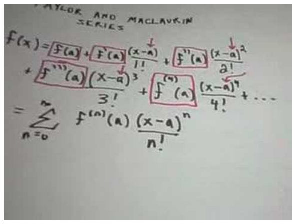
Frequently Used Maclaurin Series

Some frequently used Maclaurin Series are listed below

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	in$(-1,1)$
$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$	in$(-1,1)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	in$(-1,1)$
$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$,	in$(-1,1)$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$,	in$(-1,1)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	in$(-1,1)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	in$(-1,1)$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$	in$(-1,1)$

Multimedia Links

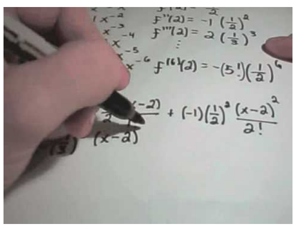
For video presentations on the Taylor and Maclaurin Series **(26.0)**, see Just Math Tutoring, Taylor and Maclaurin Series, Example1 (6:30)



MEDIA

Click image to the left for more content.

and Just Math Tutoring, Taylor and Maclaurin Series, Example2 (9:45).



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Click image to the left for more content.

8.8 Calculations with Series

Binomial Series

We have learned the Binomial Theorem for positive integer exponents:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n (BE)$$
$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

where the Binomial coefficients are denoted by

$\binom{n}{0} = 1$ and $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$ for $k \geq 1$

AS a simple Binomial function, take $a = 1$ and $b = x$. Then $(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$

Let r be a real number and $f(x) = (1+x)^r$. Is $f(x)$ equal to a series in the form of (BE) except that there may be an infinite series? The answer is yes.

Theorem (Binomial Series) Let r be a real number and $|x| < 1$. Then

$$(1+x)^r = 1 + \text{rx} + \frac{r(r-1)}{k!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3$$
$$= \sum_{k=0}^{n} \binom{r}{k} x^k$$

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$\binom{n}{0} = 1$ and $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{2!}$ for $k \geq 1$

where $\binom{r}{0} = 1$ and $\binom{r}{k} = \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k \geq 1$

Example 1 Find a power series representation of $\sqrt{1+x}$.

Solution. We need to compute the Binomial coefficients for $r = -\frac{1}{2}$

$$\binom{\frac{1}{2}}{k} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2k-3}{2}\right)}{k!} = \frac{(-1)^{k-1}1\cdot 3\cdot 5\ldots(2k-3)}{2^k k!}$$
$$= \frac{(-1)^{k-1}(2k-2)!}{2.4.6\ldots(2k-2))2^k k!} = \frac{(-1)^{k-1}(2k-2)!}{2^{k-1}k!(k-1)2^k k!} = \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!}$$

So if $|x| < 1$, $\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!} x^k$

Example 2 Find a power series representation of $\frac{1}{(1-x)^m}$ where m is a positive integer.

Solution. We need to compute the Binomial coefficients for $r=-m$ (and will replace x by $-x$)

$$\begin{aligned}\binom{-m}{k} &= \frac{(-m)(-m-1)(-m-2)\dots(-m-k+1)}{k!}\\ &= \frac{(-1)^k m(m+1)(m+2)\dots(m+k-1)}{k!} = (-1)^k\binom{m+k-1}{k}\end{aligned}$$

So

$$\begin{aligned}\frac{1}{(1-x)^m} &= \sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k}(-x)^k\\ &= \sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k}(-1)^k x^k\\ &= \sum_{k=0}^{\infty}(-1)^{2k}\binom{m+k-1}{k}x^k\\ &= \sum_{k=0}^{\infty}\binom{m+k-1}{k}x^k\end{aligned}$$

Exercises

1. Find a power series representation of $\frac{1}{\sqrt{1+x}}$ at $x=0$.
2. Find a power series representation of $\frac{1}{(2-x)^2}$ at $x=0$.
3. Notice $1+x+x^2=x+\frac{3}{4}+\left(x+\frac{1}{2}\right)^2$. Find a power series representation of $\sqrt{(1+x+x^2)}$ at $x=-\frac{1}{2}$. In what interval is the equality true?

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Then we may be able to deduce a useful Taylor Series centered close to the given x.

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Since 1.1 is close to $\frac{\pi}{3}$, we would try to find a Taylor Series of $\sin x$ at $x_0 = \frac{\pi}{3}$. Let $f(x) = \sin x$. Then

$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x$ and $f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ f'\left(\frac{\pi}{3}\right) = \frac{1}{2},\ f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},\ f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2}$.

This pattern repeats and $\lim_{n\to\infty} R_n(x) = 0$ can be checked as in the case $x_0 = 0$. So the Taylor Series is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1}$$

Hence

$$\begin{aligned}\sin(1.1) &\approx \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!}(0.0528)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!}(0.0528)^{2n+1} \\ &\approx 0.86481823 + 0.02638773 \text{ taking 2 terms from each sum} \\ &\approx 0.8912\end{aligned}$$

We may also apply algebraic manipulation to standard Taylor Series.

Example 3 Approximate $\frac{1}{1.9^2}$ to 4 decimal places.

Solution. There is standard Taylor Series:

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$\frac{1}{(2-x)^2} = \frac{1}{4}\frac{1}{(1-\frac{x}{2})^2} = \frac{1}{4}\sum_{k=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1}$ for $|x| < 2$.

So we take $x = 0.1$ and then

$$\frac{1}{1.9^2} = \frac{1}{4}\sum_{n=1}^{\infty} n\left(\frac{x}{2}\right)^{n-1} \approx \frac{1}{4}(1 + 2(0.05) + 3(0.05)^2 + 4(0.05)^3) \approx 0.2770.$$

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Hint: consider center $\frac{\pi}{4}$

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$$= e(1-u+u^2-u^3+\ldots)\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\frac{u^4}{2!}+\ldots\right)$$
$$= e\left(1+\frac{1}{2}u^2-\frac{1}{3}u^3+\frac{3}{8}u^4-\frac{11}{30}u^5+\ldots\right) \text{ for } |u|<1$$

so

$\int \frac{e^x}{x}dx = e\left(u+\frac{1}{6}u^3-\frac{1}{12}u^4+\frac{3}{40}u^5-\frac{11}{180}u^6+\ldots\right)$ where $u = x-1$

Example 2 Find the power series representation of $\int \frac{\sin x^2}{x}dx$

Solution. Direct substitution of x^2 in the Maclaurin Series of $\sin x$ gives

$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1}$ and $\frac{\sin x^2}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{4n+1}$

So

$$\int \frac{\sin x^2}{x}dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\int x^{4n+1}dx$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{x^{4n+2}}{4n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+2)}\frac{x^{4n+2}}{(2n+1)!}$$

Exercises

1. Find the power series representation (Maclaurin Series) of $\int e^{x^2}dx$ and approximate $\int_0^1 e^{x^2}dx$ to 6 decimal places.
2. Find the power series representation (Maclaurin Series) of $\int \sin x^2 dx$.

Frequently Used Maclaurin Series

Some frequently used Maclaurin Series are listed below

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$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$	in$(-1,1)$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	in$(-1,1)$
$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$,	in$(-1,1)$
$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$,	in$(-1,1)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	in$(-1,1)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	in$(-1,1)$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$	in$(-1,1)$

Texas Instruments Resources

In the CK-12 Texas Instruments Calculus FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9733 .